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An Analytical Approach to Intertrack Space Widening on Railroad Curves

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Summary

The article addresses the issue of designing the geometry of a curved double track using the analytical design method principles. This allowed, similarly to other applications of the method, a complete overview of the issue and the method for determining the key parameters to be defined. The analytical method for determining the intertrack space axis and the outer and inner track axes is introduced, leading to the required value of the intertrack space widening. The widening is achieved by varying the length of the transition curves in the outer and inner tracks. With the track axis coordinates in the local coordinate system, these can be easily transferred to the PL-2000 two-dimensional Cartesian coordinate system, i.e., an element of the national spatial reference system. The analysis continues with the issue of the chainage of axes of the intertrack space and the mainline tracks. The applicability of the proposed method and its high precision are demonstrated by examining a wide radius range of circular curves.

Keywords: railway track, widening of the intertrack space, determining the curved track axis, railway line chainage

1. Introduction

One of the railway track geometry design aspects, as addressed by works including [1–6], is track gauge widening, commonly referred to as intertrack space widening. This is performed on both straight and curved sections and usually involves moving one of the tracks laterally to gain free space on straight sections, for example, to accommodate a stop platform. In Poland, this is performed according to the principles established in the 1970s by H. Bałuch [7, 8]. These principles are still valid and are also the subject of lectures as well as scientific analyses, the work [9] from 2015 is one example. Meanwhile, other suggestions for addressing the problem also emerged in Europe in 2010 [10].

It should be noted, however, that the problem of intertrack space widening on straight sections is analogous to the case of connecting parallel tracks by means of two turnouts; this takes place along the length of the curved tracks, which adopt the shape of reverse curves. As the standard geometry of curved tracks used in railway turnouts since the dawn of railways has used a single circular curve (without transition curves), there must still be an intermediate element between the ends of the turnouts used in the track connection. However, recently, to smooth the curvature diagram, the so-called "clothoid sections" have been introduced on both sides of the circular curve where the curvature changes linearly, but often does not reach zero values at the extreme points [9, 11–14]. In this case, the intermediate element is not needed, and the end ordinate of the first turnout is equal to one half of the required track gauge. An identical second turnout is inserted into the parallel track, while the ends of both turnouts are interconnected. An even more preferable solution is the recommendation to use, in the curved track of a turnout, sections with non-linear curvature [15]. This has been demonstrated in the new intertrack space widening solution [16].

The issue of intertrack space widening on double tracks located in a circular curve must be addressed in a completely different way. In addition to the correct shaping of both track axes to achieve the required widening value, the chainage[2] determination also plays an important part. It should be determined for the intertrack space axis, which on straight sections also covers both tracks, however, on circular curves it

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[2] Chainage (mileage) – determination of a particular place on a railway line by giving its distance from the starting point of the line. Chainage in Poland has a form of an inscription on special chainage posts (milestones) placed every 100 m.

creates discrepancies for both the outer track (which is longer than the intertrack space axis) and the inner track (which is shorter). In the Technical Standards [17], this problem is not covered in particular, but there is a focus on the issue of single-track axis chainage for design purposes.

The regulations [17] generally state that track axis chainage is to be determined independently for each track of a railway line. The design chainage[3] may be assumed locally when designing the geometry of tracks and is not the same as the operational chainage of the railway line as determined at the post-execution stage.

2. Determination of intertrack space widening values

With regard to the issue of intertrack space widening, the Standards [17] stipulate that, unless there are terrain or infrastructure constraints, track systems (track axes or track connections) should be designed as parallel (on straight sections) or homocentric (on sections located within horizontal curves). Changing the gauge of mainline tracks (of significant value) shall be performed according to the principles defined in the work [8]. Apart from this, the track gauge must comply with the structure gauge requirements defined under separate regulations.

Pursuant to Technical Conditions [18], as part of maintenance on the PKP Polskie Linie Kolejowe S.A. network, the GPL-1 standardised structure gauge should be observed (when the existing structure location permits this, and operational parameters of the line or section do not stipulate otherwise). Compliance with this gauge ensures that normative national gauges are met. This gauge shall be assessed separately for the part up to 1170 mm high and for the part above it.

In the case of the GPL-1 standardised structure gauge, the additional clearance specified in the Structure Gauge Type Sheet [19] is required. Each sheet covers a nominal clearance structure gauge and the extreme structure gauge in an explanatory form, a table of coordinates of points located on these clearance gauges and a list of formulas accounting for the effect of horizontal curvature (Δb_s). All clearance gauges shall be canted in the presence of track design cant, whereby the effect of cant (Δb_D) is determined by the horizontal component of the displacement of the considered point of the clearance gauge due to cant.

The clearance gauge rules also apply directly to the track gauge. In the case of non-built-up intertrack space, the track gauge is to be referred to the twofold width of half of the clearance gauges indicated on the Structure Gauge Type Sheet. In this case, the clearance requirements at the standardised structure gauge do not apply.

The regulations [18] also include – as optional (and therefore further applicable) – Annex III: Structure gauge on straight and curved track sections (according to the rules applicable before the introduction of the A2 module). Formulas for the values of the increased nominal horizontal gauge dimensions on a curve are included there:

- in the inner part of the curve:

$$\Delta b_w = \Delta b_R + \Delta b_h, \tag{1}$$

- in the outer part of the curve:

$$\Delta b_z = \Delta b_R, \tag{2}$$

where:
Δb_R – the widening caused by the positioning of the vehicle along the chord,
Δb_h – the widening caused by tilting of the body of a railroad vehicle.

The widening values of the horizontal gauge dimensions Δb_R depend on the radius of a circular curve. The relevant table indicates that for $R > 500$ m they are relatively small and amount to several dozen millimeters. Changes to the gauge due to lateral tilting of a vehicle body on a canted track shall be considered as a widening of the horizontal dimensions of the gauge only on the inner side of the curve. With a specified cant h, the characteristic points of the gauge at a height H_i above the inner rail head are displaced inwards into the curve by a value of:

$$\Delta b_h = \frac{H_i h}{\sqrt{1500^2 - h^2}}. \tag{3}$$

The presented rules for widening the structure gauge on curves determine the required values of intertrack space widening on double tracks located in a circular curve. The following formula should be applied here:

$$\Delta b_m = 2\Delta b_R + \max\Delta b_h, \tag{4}$$

[3] Design chainage (local) – identifying a specific location on a railway line by providing its distance from a selected specific point (e.g., the axis of the adjacent station). It should be in line with the line chainage growth direction.

where by the value maxΔb_h should result from the cant h_{max} = 150 mm and the height H_i = 4850 mm.

In this situation, the intertrack space d_m of double tracks located on curved railroads is determined according to the formula:

$$d_m = d_0 + \Delta b_m, \quad (5)$$

where: d_0 is the nominal track gauge for straight sections.

The article addresses the issue of designing the geometry of a curved double track using the analytical design method principles [20, 21]. This provided a complete overview of the considered problem and defined how the key parameters should be determined. The analysis continued with the issue of railway line chainage.

An analysis of an elementary (symmetrical) geometry composed of a circular curve and two transition curves of the same type and length has been assumed. The course of action adopted in the article is as follows:

- analytical record of the intertrack space axis,
- determination of outer track parameters,
- determination of inner track parameters,
- determination of the difference between ordinates in the local coordinate system,
- transformation of the obtained solution to the PL-2000 system [22],
- establishing the chainage for the axis of the intertrack space and the mainline tracks.

Analogously to the works [20, 21], we adopt a local coordinate system (*LCS*) linked to the intertrack space axis in the given case. The axes of both tracks will also be analytically noted in this system.

3. Analytical notation of the intertrack space axis

The basic parameters of the geometry considered for the intertrack space are the radius R_m of the circular curve of the intertrack space axis and the length l_m of the transition curve on the intertrack space axis. When selecting these parameters, it must be noted that the corresponding values in the outer (R_z and l_z) and inner (R_w and l_w) tracks are different; they too – as well as the values of R_m and l_m – must comply with the relevant kinematic conditions. A transition curve is assumed in the form of a clothoid with curvature as described by the formula:

$$k(l) = \frac{1}{R_m l_m} l, \quad (6)$$

and the inclination angle of the tangent:

$$\Theta(l) = \frac{1}{2R_m l_m} l^2, \quad (7)$$

where: the linear coordinate l determines the position of a given point of the curve along its length.

The parametric equations of the K_{Pm} (clothoid) curve in the x_m, y_m coordinate system associated with this curve are as follows [20]:

$$x_m(l) = l - \frac{1}{40R_m^2 l_m^2} l^5 + \frac{1}{3456R_m^4 l_m^4} l^9 - \frac{1}{599040R_m^6 l_m^6} l^{13}, \quad (8)$$

$$y_m(l) = -\frac{1}{6R_m l_m} l^3 + \frac{1}{336R_m^3 l_m^3} l^7 - \frac{1}{42240R_m^5 l_m^5} l^{11}. \quad (9)$$

At the end of the curve:

$$x_m(l_m) = l_m - \frac{l_m^3}{40R_m^2} + \frac{l_m^5}{3456R_m^4} - \frac{l_m^7}{599040R_m^6}, \quad (10)$$

$$y_m(l_m) = -\frac{l_m^2}{6R_m} + \frac{l_m^4}{336R_m^3} - \frac{l_m^6}{42240R_m^5}, \quad (11)$$

$$Q_m(l_m) = -\frac{l_m}{2R_m}. \quad (12)$$

For a curve deflection angle α, the transformation of the clothoid to the adopted local coordinate system (shown in Figure 1) is achieved by rotating the x_m, y_m system clockwise by the angle $\alpha/2$. This operation results in the parametric equations of the curve in *LCS*:

$$x(l) = x_m(l)\cos\frac{\alpha}{2} - y_m(l)\sin\frac{\alpha}{2}, \quad (13)$$

$$y(l) = x_m(l)\sin\frac{\alpha}{2} + y_m(l)\cos\frac{\alpha}{2}, \quad (14)$$

The parameter $l \in \langle 0, l_m \rangle$ and the abscissa of the transition curve $x \in \langle 0, x(l_m) \rangle$ appear in these equations, where:

$$x(l_m) = x_m(l_m)\cos\frac{\alpha}{2} - y_m(l_m)\sin\frac{\alpha}{2} = L_{KPm}. \quad (15)$$

The terminal ordinate of the transition curve is:

$$y(l_m) = x_m(l_m)\sin\frac{\alpha}{2} + y_m(l_m)\cos\frac{\alpha}{2} = y_{KPm}, \quad (16)$$

whereas the inclination angle of the tangent is:

$$\Theta(l_m) = -\frac{l_m}{2R_m} + \frac{\alpha}{2}, \quad (17)$$

hence the value of the tangent inclination is:

$$s(l_m) = \tan\Theta(l_m) = \tan\left(-\frac{l_m}{2R_m} + \frac{\alpha}{2}\right) = s_{KPm}. \quad (18)$$

Given the position of the transition curve, a circular curve of radius R_m can be plotted in the geometry. The length of half of its projection on the x-axis, i.e., the value of L_{LKm}, is determined by the tangency:
- at the origin of the curve (i.e., for $x = L_{KPm}$) $s = s_{KPm}$,
- in the midpoint of the curve (i.e., for $x = L_{KPm} + L_{ŁKm}$) $s = 0$,

from which it follows that:

$$L_{ŁKm} = \frac{s_{KPm}}{\sqrt{1 + s_{KPm}^2}} R_m. \quad (19)$$

Knowing L_{KPm} and $L_{ŁKm}$ enables notation of the circular curve equation as an explicit function $y = y(x)$.

$$y(x) = y_{KPm} + \sqrt{R_m^2 - \left[L_{KPm} + L_{LKm} - x\right]^2} - \sqrt{R_m^2 - L_{LKm}^2},$$

$$x \in \langle L_{KPm}, L_{KPm} + L_{LKm} \rangle. \quad (20)$$

By denoting the abscissa of the midpoint $L_{KPm} + L_{ŁKm} = x_{Sm}$, the following formula for the ordinate of the curve is given:

$$y(x_{Sm}) = y_{KPm} + R_m - \sqrt{R_m^2 - L_{LKm}^2} = y_{Sm}. \quad (21)$$

Due to symmetry, the local coordinate system covered half of the entire system, i.e., the region from the origin of the transition curve to the midpoint of the circular curve. It is still required to complete the ordinates for the second part of the designed area, i.e., for: $x \in \langle L_{KPm} + L_{LKm}, 2L_{KPm} + 2L_{LKm} \rangle$. They will be a mirror image of the presented solution achieved for $x \in \langle 0, L_{KPm} + L_{LKm} \rangle$.

For the second half of the circular curve, formula (20) applies, as used for the first half of the curve. For the second transition curve, i.e., for $x \in \langle L_{KPm} + 2L_{LKm}, 2L_{KPm} + 2L_{LKm} \rangle$, parametric equation (14) and another equation for $x(l)$ apply:

$$x(l) = 2x(l_m) + 2L_{LKm} - \left[x_m(l)\cos\frac{\alpha}{2} - y_m(l)\sin\frac{\alpha}{2}\right], \quad (22)$$

where: $x_m(l)$ follows from formula (8) and $y_m(l)$ from formula (9), where $l \in \langle 0, l_m \rangle$.

The presented theoretical relationships have been applied in the calculation example with the obligatory curve deflection angle $\alpha = \pi/2$ rad, in which, for the speed of trains $V = 120$ km/h, the value of the radius $R_m = 900$ m, the cant on the curve $h_0 = 80$ mm and the length of the transition curve (clothoid) $l_m = 115$ m have been assumed. The selection of the l_m value involved the necessity of adopting a shorter curve in the outer track (in connection with the requirements con-

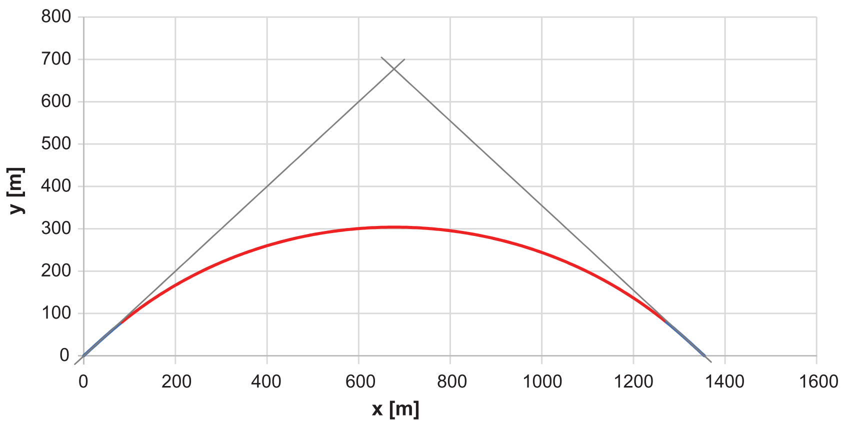


Fig. 1. The geometry of the intertrack space axis in the local coordinate system; red indicates the circular curve, blue the transition curves (curve deflection angle $\alpha = \pi/2$ rad, curve radius $R_m = 900$ m, clothoid length $l_m = 115$ m) [author's study]

cerning the widening of the intertrack space), which must also meet the kinematic conditions. Figure 1 shows the geometric solution obtained.

4. Determination of the outer track axis

The radius of the circular curve of the outer track, being concentric with the axis of the intertrack space, is $R_z = R_m + \frac{d_m}{2}$. The ordinate of the midpoint of the circular curve in the outer track is also increased by a value of $d_m/2$ in relation to the corresponding ordinate of the circular curve in the intertrack space axis:

$$y_{Sz} = y_{Sm} + \frac{\Delta b_m}{2}. \quad (23)$$

The origin of the transition curve in the outer track is located at point O_z located in the main line of this track described by the equation:

$$y = \left(\tan\frac{\alpha}{2}\right)x + \frac{d_0}{2\cos\frac{\alpha}{2}}. \quad (24)$$

The coordinates of the origin of the transition curve are not known at this stage of the procedure. Equation (24) shows that they are linked by the relationship:

$$y_{Oz} = \left(\tan\frac{\alpha}{2}\right)x_{Oz} + \frac{d_0}{2\cos\frac{\alpha}{2}}. \quad (25)$$

The key value determining the potential for widening the intertrack space is the length l_z of the transition curve in the outer track. This is because, for the track in question, all the relationships concerning the intertrack space axis apply, i.e., equations (6)÷(22), in which the length l_m has been replaced by l_z.

The value of l_z must be selected so that the ordinate of the midpoint y_{Sz} of the circular curve meets condition (23). Assuming the analogy to equation (21), the following equation is obtained:

$$y_{KPz} + R_z - \sqrt{R_z^2 - L_{LKz}^2} = y_{Sm} + \frac{d_m}{2},$$

i.e.:

$$y_{Oz} + x_z(l_z)\sin\frac{\alpha}{2} + y_z(l_z)\cos\frac{\alpha}{2} + R_z - \sqrt{R_z^2 - L_{LKz}^2} = y_{Sm} + \frac{d_m}{2}. \quad (26)$$

Meeting condition (26) entails assuming an appropriate value of l_z and finding y_{Oz}, which will belong to the base line. To determine both unknowns, it is still necessary, by analogy with equation (15), to consider the abscissa compatibility condition.

$$x_{Oz} + x_z(l_z)\cos\frac{\alpha}{2} - y_z(l_z)\sin\frac{\alpha}{2} + L_{LKz} = x_{Sm}. \quad (27)$$

From the system of equations (26) and (27), after considering relationship (25) and appropriate transformations, the following expression can be obtained:

$$\frac{d_0}{2\cos\frac{\alpha}{2}} - \left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\tan\frac{\alpha}{2}\right)\left(\frac{l_z^2}{6R_z} - \frac{l_z^4}{336R_z^3} + \frac{l_z^6}{42240R_z^5}\right) + \left(1 - \frac{1 + \tan\frac{\alpha}{2}\tan\left(-\frac{l_z}{2R_z} + \frac{\alpha}{2}\right)}{\sqrt{1 + \left[\tan\left(-\frac{l_z}{2R_z} + \frac{\alpha}{2}\right)\right]^2}}\right)R_z = y_{Sm} + \frac{d_m}{2} - \left(\tan\frac{\alpha}{2}\right)x_{Sm} \quad (28)$$

from which the length l_z of the transition curve in the outer track is iteratively determined.

In the next step, using equation (27), the abscissa x_{Oz} of the origin of the transition curve in the outer track is determined.

$$x_{Oz} = x_{Sm} - \left(l_z - \frac{l_z^3}{40R_z^2} + \frac{l_z^5}{3456R_z^4} - \frac{l_z^7}{599040R_z^6}\right)\cos\frac{\alpha}{2} - + \left(\frac{l_z^2}{6R_z} - \frac{l_z^4}{336R_z^3} + \frac{l_z^6}{42240R_z^5}\right)\sin\frac{\alpha}{2} - + \frac{\tan\left(-\frac{l_z}{2R_z} + \frac{\alpha}{2}\right)}{\sqrt{1 + \left[\tan\left(-\frac{l_z}{2R_z} + \frac{\alpha}{2}\right)\right]^2}}R_z. \quad (29)$$

With this data, a further procedure can be initiated, analogous to that for the intertrack space axis. In the calculation example, the value of the required intertrack space widening first had to be determined using formula (4). The regulations [18], imply that for the radius R_m = 900 m the value Δb_R is 40 mm, while for the cant on the curve h_0 = 80 mm the widening maxΔb_h = 260 mm. On this basis, the widening of the intertrack space Δb_m = 340 mm. This means that, with a nominal track gauge of $d_0 = 4.00$ m, the distance between the track axes (outer and inner) on the circular curve in the case in question is $d_m = 4.34$ m. The value of the radius of the circular curve in the outer track is $R_z = 902.17$ m.

The length l_z of the transition curve in the outer track was determined iteratively from equation (28). The relevant procedure is shown in Table 1. The values of $x_z(l_z)$ were obtained using a suitably modified formula (8). Analogously, formula (9) has been used to determine $y_z(l_z)$, formula (18) for s_{KPz} and formula (19) for $L_{ŁKz}$. In addition, the notations in Table 1 have been adopted:

$$\frac{d_0}{2\cos\frac{\alpha}{2}} + \left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\tan\frac{\alpha}{2}\right) y_z\left(l_z\right) + R_z - $$

$$+\sqrt{R_z^2 - L_{ŁKz}^2} - \left(\tan\frac{\alpha}{2}\right) L_{ŁKz} = \sum L,$$

$$y_{Sm} + \frac{d_m}{2} - \left(\tan\frac{\alpha}{2}\right) x_{Sm} = \sum P,$$

where: the values x_{Sm} and y_{Sm} are derived from the calculations for the intertrack space axis and are: $x_{Sm} = 677.4821$ m, $y_{Sm} = 303.8241$ m.

According to Table 1, the conditions of the problem are met by the value $l_z = 97.7705$ m. The abscissa of the origin of the first transition curve, as determined by formula (29), $x_{Oz} = 4.675$ m, and the ordinate $y_{Oz} = 7.503$ m as determined by formula (25). Figure 2 shows the initial region of the obtained geometrical solution for the outer track.

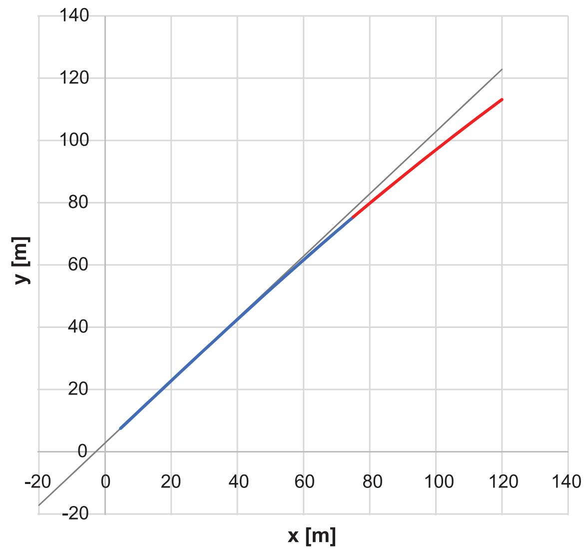


Fig. 2. The initial region of the geometrical solution for the outer track in the local coordinate system; red is the fragment of the circular curve, and blue is the transition curve (curve deflection angle $\alpha = \pi/2$ rad, curve radius $R_z = 902.17$ m, clothoid length $l_z = 97.7705$ m) [author's study]

Table 1

Iterative process for determining the length of the transition curve in the outer track

l_z [m]	$x_z(l_z)$ [m]	$y_z(l_z)$ [m]	s_{KPz}	$L_{ŁKz}$ [m]	ΣL [m]	ΣL [m]	$\Sigma P - \Sigma L$ [m]
100	99.969	–1.848	0.894875	672.288	–371.517	–371.488	0.028892407
99	98.970	–1.811	0.895873	671.955	–371.504	–371.488	0.015852305
98	97.971	–1.775	0.896873	671.621	–371.491	–377.488	**0.002944098**
97	96.972	–1.739	0.897873	671.287	–371.478	–377.488	**–0.00983224**
96	95.973	–1.703	0.898875	670.953	–371.465	–377.488	–0.02247673
95	94.974	–1.668	0.899877	670.619	–371.453	–377.488	–0.03498941
97.9	97.871	–1.771	0.896973	671.588	–371.490	–377.488	0.00166053
97.8	97.771	–1.767	0.897073	671.554	–371.488	–371.488	**0.000378281**
97.7	97.671	–1.764	0.897173	671.521	–371.487	–371.488	**–0.00090265**
97.6	97.571	–1.760	0.897273	671.487	–371.486	–371.488	0.00218226
97.79	97.761	–1.767	0.897083	671.551	–371.488	–371.488	0.000250129
97.78	97.751	–1.767	0.897093	671.548	–371.488	–371.488	**0.00012199**
97.77	97.741	–1.766	0.897103	671.544	–371.488	–371.488	**–6.1361E–06**
97.76	97.731	–1.766	0.897113	677.541	–377.488	–371.488	–0.00013425
97.772	97.743	–1.766	0.897101	671.545	–377.488	–377.448	1.9488E–05
97.771	97.742	–1.766	0.897102	671.545	–377.488	–377.488	6.67591E–06
97.7705	97.742	–1.766	0.897102	671.544	–377.488	–377.488	**2.69889E–07**
97.770	97.741	–1.766	0.897103	671.544	–371.488	–371.488	–6.1361E–06
97.769	97.740	–1.766	0.897104	671.544	–371.488	–371.488	–1.8948E–05

[Author's study].

5. Determination of the inner track axis

The radius of the circular curve of the inner track, being concentric with the axis of the intertrack space, is $R_w = R_m - \frac{d_m}{2}$. The ordinate of the midpoint of the circular curve in the inner track is decreased by a value of $d_m/2$ in relation to the corresponding ordinate of the circular curve in the intertrack space axis according to the formula:

$$y_{Sw} = y_{Sm} - \frac{d_m}{2}. \quad (30)$$

The origin of the transition curve in the inner track is located at point O_w located in the main line of this track described by the equation:

$$y = \left(\tan\frac{\alpha}{2}\right)x - \frac{d_0}{2\cos\frac{\alpha}{2}}. \quad (31)$$

The coordinates of the start of the transition curve are not known. Equation (31) shows that they are linked by the relationship:

$$y_{Ow} = \left(\tan\frac{\alpha}{2}\right)x_{Ow} - \frac{d_0}{2\cos\frac{\alpha}{2}}. \quad (32)$$

As with the outer track, the value that determines the potential for widening the intertrack space is the length l_w of the transition curve in the inner track. The relationships concerning the intertrack space axis also apply for this track, i.e., equations (6)÷(22), in which the length l_m has been replaced by l_w.

The value of l_w must be selected so that the ordinate of the midpoint y_{sw} of the circular curve meets condition (30). Assuming the analogy to equation (21), the following equation is obtained:

$$y_{KPw} + R_w - \sqrt{R_w^2 - L_{LKw}^2} = y_{Sm} - \frac{d_m}{2},$$

i.e.:

$$y_{Ow} + x_w(l_w)\sin\frac{\alpha}{2} + y_w(l_w)\cos\frac{\alpha}{2} + R_w - \\ + \sqrt{R_w^2 - L_{LKw}^2} = y_{Sm} - \frac{d_m}{2}. \quad (33)$$

Meeting condition (33) entails taking the appropriate value of l_w and finding y_{Ow}, which will belong to the base line; to determine both unknowns, it is still necessary, taking into account the analogy with equation (15), to consider the abscissa compatibility condition.

$$x_{Ow} + x_w(l_w)\cos\frac{\alpha}{2} - y_w(l_w)\sin\frac{\alpha}{2} + L_{LKw} = x_{Sm}. \quad (34)$$

From the system of equations (33) and (34), after considering relationship (32) and appropriate transformations, the following expression is obtained:

$$-\frac{d_0}{2\cos\frac{\alpha}{2}} - \left(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\tan\frac{\alpha}{2}\right)\left(\frac{l_w^2}{6R_w} - \frac{l_w^4}{336R_w^3} + \frac{l_w^6}{42240R_w^5}\right) + \\ + \left(1 - \frac{1 + \tan\frac{\alpha}{2}\tan\left(-\frac{l_w}{2R_w} + \frac{\alpha}{2}\right)}{\sqrt{1 + \left[\tan\left(-\frac{l_w}{2R_w} + \frac{\alpha}{2}\right)\right]^2}}\right)R_w = y_{Sm} - \frac{d_m}{2} - \left(\tan\frac{\alpha}{2}\right)x_{Sm}. \quad (35)$$

from which the length l_w of the transition curve in the inner track can be iteratively determined. In the next step, using equation (34), the abscissa x_{Ow} of the origin of the transition curve in the inner track is determined.

$$x_{Ow} = x_{Sm} - \left(l_w - \frac{l_w^3}{40R_w^2} + \frac{l_w^5}{3456R_w^4} - \frac{l_w^7}{599040R_w^6}\right)\cos\frac{\alpha}{2} - \\ + \left(\frac{l_w^2}{6R_w} - \frac{l_w^4}{336R_w^3} + \frac{l_w^6}{42240R_w^5}\right)\sin\frac{\alpha}{2} - \\ + \frac{\tan\left(-\frac{l_w}{2R_w} + \frac{\alpha}{2}\right)}{\sqrt{1 + \left[\tan\left(-\frac{l_w}{2R_w} + \frac{\alpha}{2}\right)\right]^2}}R_w. \quad (36)$$

The further procedure is the same as for the intertrack space and outer track axes. In the calculation example, the value of the circular curve radius in the inner track is $R_w = 897.83$ m. The length l_w of the transition curve in the inner track was determined iteratively from equation (35) using the same procedure as in Table 1. It was found that the conditions of the problem are fulfilled by the value $l_w = 129.636$ m. The abscissa of the origin of the first transition curve x_{Ow} determined by formula (36) = −3.756 m, and the ordinate y_{Ow} determined by formula (32) $y_{Ow} = -6.585$ m. Figure 3 shows the initial region of the geometry of the considered double track. It illustrates the design of the intertrack space widening that takes place along the length of the transition curves.

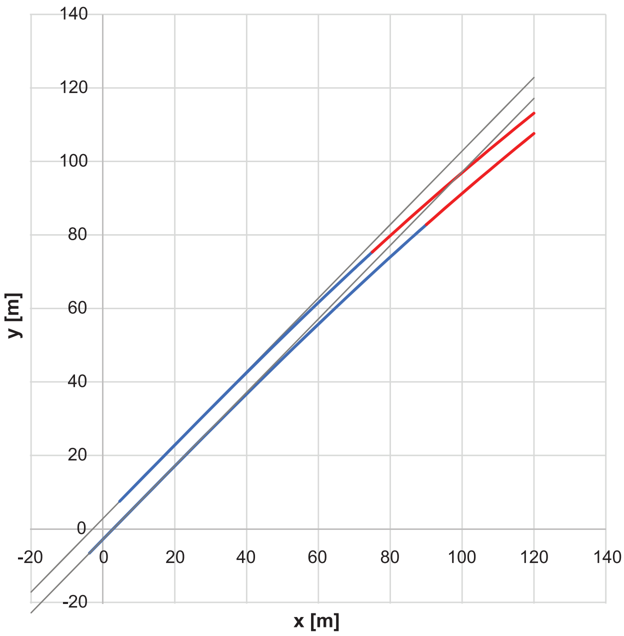


Fig. 3. The geometry of the considered double track in the initial section; fragments of circular curves are marked in red and transition curves in blue (curve deflection angle α = $\pi/2$ rad, curve radii R_z = 902.17 m and R_w = 897.83 m, clothoid lengths l_z = 97.7705 m and l_w = 129.636 m) [author's study]

6. Differences between ordinates in the local coordinate system

To be able to compare the values of ordinates of the outer track and inner track axes against the intertrack space axis, the initial point *P* of the straight-line section to be analysed shall be determined ahead of the curve; the end point *K* will adopt a symmetrical position. The corresponding coordinates of the intertrack space axis x_{Pm} = −20 m and y_{Pm} = −20 m have been adopted as reference values. This corresponds to the coordinates of the point located on the straight behind the curve x_{Km} = 1374.964 m and y_{Km} = −20 m.

The coordinates of the initial point in the outer and inner track are derived from the values of the coordinates of point *P* on the intertrack space axis. They are calculated using the following formulas:

$$x_{Pz} = \frac{\tan\frac{\alpha}{2}}{1+\left(\tan\frac{\alpha}{2}\right)^2}\left(\frac{1+\tan\frac{\alpha}{2}}{\tan\frac{\alpha}{2}}\,y_{Pm} - \frac{d_0}{2\cos\frac{\alpha}{2}}\right), \quad (37)$$

$$y_{Pz} = y_{Pm} - \frac{1}{\tan\frac{\alpha}{2}}\left(x_{Pz} - x_{Pm}\right), \quad (38)$$

for the outer track, and for the inner track from the formulas:

$$x_{Pw} = \frac{\tan\frac{\alpha}{2}}{1+\left(\tan\frac{\alpha}{2}\right)^2}\left(\frac{1+\tan\frac{\alpha}{2}}{\tan\frac{\alpha}{2}}\,y_{Pm} + \frac{d_0}{2\cos\frac{\alpha}{2}}\right), \quad (39)$$

$$y_{Pw} = y_{Pm} - \frac{1}{\tan\frac{\alpha}{2}}\left(x_{Pw} - x_{Pm}\right). \quad (40)$$

In the calculation example in question, the following values are obtained for the outer track: x_{Pz} = −21.414 m, y_{Pz} = −18.586 m and the corresponding (by symmetry) x_{Kz} = 1376.378 m, y_{Kz} = −18.586 m. For the inner track, the analogous values are as follows: x_{Pw} = −18.586 m, y_{Pw} = −21.414 m, x_{Kw} = 1373.550 m, y_{Kw} = −21.414 m. Figures 4 and 5 show the track and intertrack space axes for the initial zone and the end zone.

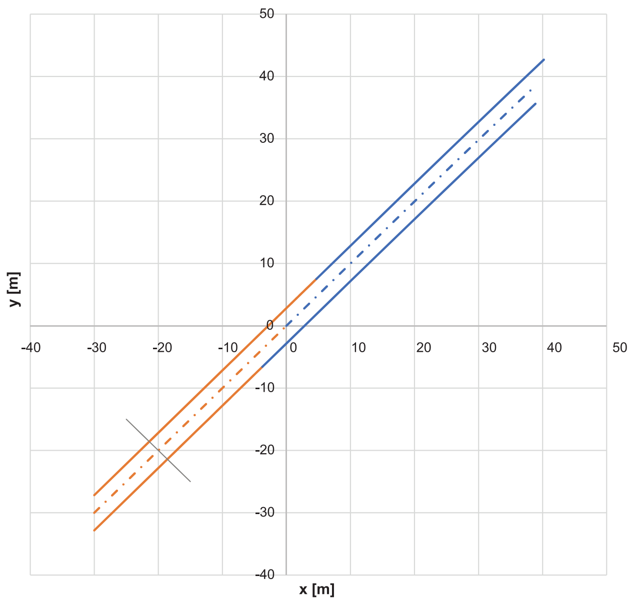


Fig. 4. Geometries of the outer (top) and inner (bottom) track axes, as well as the intertrack space axis (midpoint) for the origin zone; blue indicates the transition curve sections (curve deflection angle $\alpha = \pi/2$ rad, curve radii R_z = 902.17 m, R_m = 900 m and R_w = 897.83 m, clothoid lengths l_z = 97.7705 m, l_m = 115 m, l_w = 129.636 m) [author's study]

Figure 6 illustrates a graph of the differences between ordinates of the outer track axis and the inner track axis with respect to the ordinates of the intertrack space axis along the length of the geometry in question. As illustrated, because of the inclination angle of both directions of the main routes $\alpha/2$ the determined value of the track gauge in the circular curve d_m = 4.34 m occurs only in the midpoint of the entire system, for x_S = 677.482 m. The difference between ordinates increases in the direction of both the start and the end of the system to assume a constant value equal to $d_0/(\cos\alpha/2)$, i.e., in this case 5.657 m.

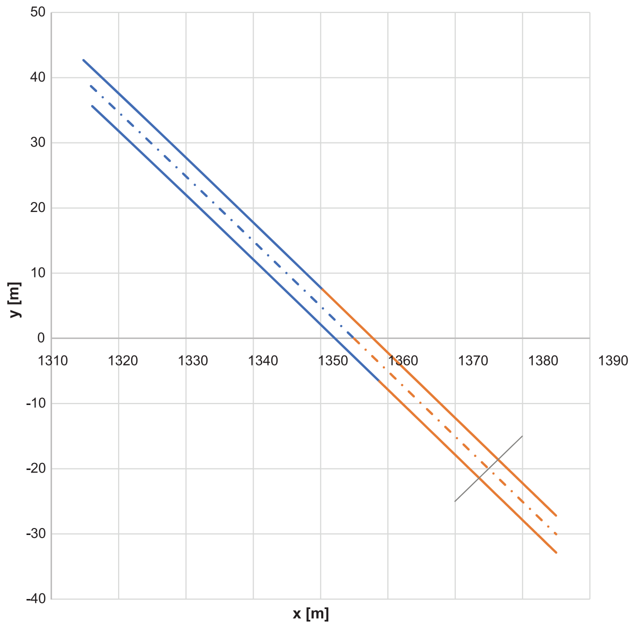


Fig. 5. Geometries of the outer (top) and inner (bottom) track axes, as well as the intertrack space axis (midpoint) for the end zone; blue indicates the transition curve sections (curve deflection angle α = π/2 rad, curve radii R_z = 902.17 m, R_m = 900 m and R_w = 897.83 m, clothoid lengths l_z = 97.7705 m, l_m = 115 m, l_w = 129.636 m) [author's study]

7. Transformation of the obtained solution to the PL-2000 system

Transferring the *x*, *y* track axis coordinates designed in *LCS* to the PL-2000 two-dimensional Cartesian coordinate system (part of the national spatial reference system [22]) is achieved using the following formulas [23]:

$$Y = Y_0 + x\cos\beta - y\sin\beta, \tag{41}$$

$$X = X_0 + x\sin\beta + y\cos\beta, \tag{42}$$

where: Y_0 and X_0 are the coordinates of the origin of the local coordinate system in the PL-2000 system, while β is the angle of rotation of the global system that is required to achieve symmetrical alignment of the cardinal directions.

Assuming a transformation of the intertrack space axis to the PL-2000 system, the following values are adopted to be the link of the *LCS* to the PL-2000 system:

- the coordinates of the point of intersection of the cardinal directions:

$$Y_W = 6512672.516 \text{ m}, X_W = 6016847.921 \text{ m},$$

- angle of rotation β = 0.76094442 rad.

Since, in the considered calculation case, the coordinates of the intersection point of the cardinal directions in the *LCS* are: x_W = 677.482 m and y_W = 677.482 m, the coordinates of the origin of the local coordinate system in the PL-2000 system can be easily determined. Their values are as follows: Y_0 = 6512649.089 m and X_0 = 6015890.103 m.

Figure 7 illustrates the geometry of the intertrack space axis in the PL-2000 global coordinate system, obtained by transforming the *x*, *y* coordinates using formulas (41) and (42).

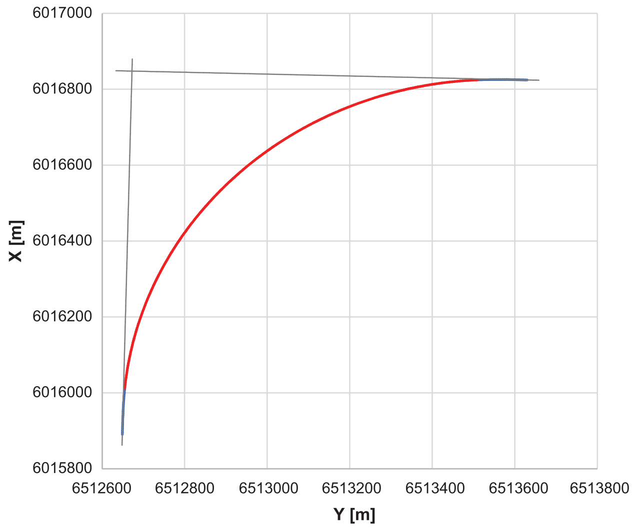
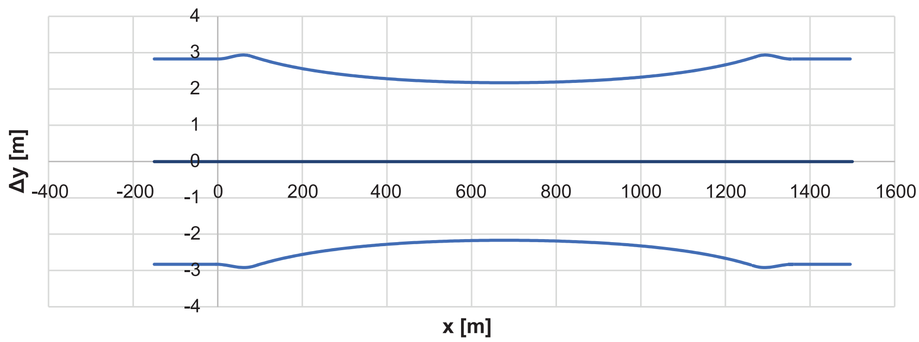


Fig. 7. The geometry of the intertrack space axis from Figure 1 in the local coordinate system PL-2000; red indicates the circular curve, blue the transition curves (curve deflection angle α = π/2 rad, curve radius R_m = 900 m, clothoid length l_m = 115 m) [author's study]

Fig. 6. The graph of differences between ordinates of the outer track axis and the inner track axis with respect to the ordinates of the intertrack axis along the length of the geometry in question (curve deflection angle α = π/2 rad, curve radii R_z = 902.17 m, R_m = 900 m and R_w = 897.83 m, lengths of the clothoid lz = 97.7705 m, l_m = 115 m, l_w = 129.636 m) [author's study]



8. Chainage of the axes of the intertrack space and the mainline tracks

Direct application of the regulations [18] concerning the principles for determining track axis chainage would, on a double track, lead to differences in the length of both tracks due to the presence of sections located in a circular curve. It seems that, in this case, the operational chainage of the railway line should run along the axis of the intertrack space, while on curves for both tracks, the design chainage should be used. This is the assumption made in this section. In the design documentation, the determination of chainage consists in determining the coordinates of the occurring characteristic points in the Cartesian coordinate system.

If a mathematical notation of the intertrack space and track axes is available, then for straight sections and transition curves this does not pose any problem as the given values of ordinates and abscissae can be used directly, while for sections located on a circular curve, knowing their lengths Δl, it is possible to determine the coordinates of their extreme points. It is necessary first to calculate the corresponding values of the central angle from the formula:

$$\Delta\Theta = \frac{\Delta l}{R}. \tag{43}$$

The key value is the difference $\Theta - \Delta\Theta$, where Θ is the central angle between the origin of the circular curve and its midpoint, with the curve aligned symmetrically. For points on the left-hand side of a circular curve (i.e., for $x_i < x_S$), the following formulas apply:

$$x_i = x_S - R\sin\left(\Theta - \Delta\Theta\right), \tag{44}$$

$$y_i = y_S + R\cos\left(\Theta - \Delta\Theta\right). \tag{45}$$

For points to the right of the circular curve (i.e., for $x_i > x_S$), the formula changes to x_i:

$$x_i = x_S + R\sin\left(\Theta - \Delta\Theta\right). \tag{46}$$

In the calculation example, the origin of the chainage of the intertrack space and track axes, i.e., the value $L = 0$, was assumed at the points corresponding to the position of the point $P_m = (-20, -20)$ m in Figure 4. For individual characteristic points (including hectometric posts), the values of the linear coordinate L, the coordinates x, y of *LCS* and Y, X coordinates of the PL-2000 system have been determined. This data provides an easy way to determine these points in the field. Table 2 provides the results of calculations for the intertrack space axis (useful for designing line chainage).

The numerical values in Table 2 for the origin and end of the system, which correspond to the intertrack space axis, can be integrated into the railway line chainage. This also applies to the origins and ends of both transition curves and the midpoint of the circular curve.

Table 3 summarises the location of the standard geometric parameters of the geometry in question. Tables 2 and 3 indicate that the total length of the system for the outer track axis, at 1,588.690 m, is 3.405 m greater than the length of the intertrack space axis; the latter in turn is 3.397 m greater than the length of the system for the inner track axis, which is 1,581.888 m.

9. Assessment of the proposed method applicability

The proposed method for determining intertrack space widening parameters is presented on the example of the selected geometry, covering the curve deflection area with a circular curve of radius $R_m = 900$ m (it was associated with the assumed train running speed $V = 120$ km/h). To assess the applicability of this method for other geometric situations, a detailed analysis was performed for two further cases characterised by significantly different curve radius values: $R_m = 300$ m (permitting a passage at $V = 80$ km/h) and $R_m = 2000$ m (for running at the speed of $V = 180$ km/h). The geometries of the relevant curve deflection areas are shown in Figure 8, while a list of the calculated parameters for the intertrack space, outer and inner track axes is given in Table 4.

From the figures in Table 4, it is clear that the proposed method is correct. For each curve radius R_m on the intertrack space, the abscissa of the curve midpoint in the outer track x_{Sz} and the inner track x_{Sw} is the same as the corresponding abscissa for the intertrack space x_{Sm} (i.e., $x_{Sz} = x_{Sw} = x_{Sm}$). Crucial for the verification of the calculations, however, is the intertrack space width d_m determined by the formula:

$$d_m = y_{Sz} - y_{Sw}, \tag{47}$$

as the difference between ordinates of the curve midpoint – in the outer track y_{Sz} and in the inner track y_{Sw}. As shown, the values of d_m given in Table 4 closely correspond to the adopted values of the intertrack space widening Δb_m. The overall compilation presented demonstrates the absolute precision of the calculations applied. It can therefore be assumed that the proposed method of determining the intertrack

Table 2

Determined coordinates of the characteristic points of the intertrack space axis (R_m = 900 m, l_m = 115 m)

Characteristic points	*L* [m]	*x* [m]	*y* [m]	*Y* [m]	*X* [m]
Origin of the system	0	20.000	-20.000	6,512,648.397	6,015,861.827
Origin of *KP1*	28.284	0.000	0.000	6,512,649.089	6,015,890.103
	100	51.128	50.288	6,512,651,436	6,015,961.779
End of *KP1*	143.284	83.015	79.553	6,512,654.347	6,016,004.962
	200	126.750	115.649	6,512,661.127	6,016,061.262
	300	209.075	172.326	6,512,681.661	6,016,159.078
	400	297.177	219.525	6,512,712.914	6,016,254.015
	500	389.969	256.664	6,512,754.502	6,016,344.901
	600	486.307	283.285	6,512,805.910	6,016,430.614
	700	585.003	299.060	6,512,866.506	6,016,510.100
Midpoint of *ŁK*	792.642	677.482	303.824	6,512,930.193	6,016,577.323
	800	684,840	303.794	6,512,935.542	6,016,582.375
	900	784.585	297.429	6,513,012.166	6,016,646.551
	1000	883.010	280.042	6,513,095.433	6,016,701.834
	1100	978.900	251.850	6,513,184.317	6,016,747.544
	1200	1071.073	213.198	6,513,277.721	6,016,783.116
	1300	1158.391	164.565	6,513,374.494	6,016,808.112
	1400	1239.779	106.549	6,513,473.441	6,016,822.223
End of *KP2*	1442.001	1271.949	79.553	6,513,515.355	6,016,824.857
	1500	1314.449	40.094	6,513,573.344	6,016,825.590
Origin of *KP2*	1557.001	1354.964	0.000	6,513,630.334	6,016,824.494
End of the system	1585.285	1374.964	–20.000	6,513,658.609	6,016,823.802

[Author's study].

Table 3

A list of locations of the main geometric parameters of the geometry in question

Geometric parameter [m]	Outer track	Intertrack space	Inner track
Radius of *ŁK*	902.17	900	897.83
Length of *KP* (clothoid)	97.772	115	129.636
The chainage of the system origin	0	0	0
Initial abscissa x_P	21.414	20.000	18.586
Initial ordinate y_P	18.586	20.000	21.414
Initial abscissa Y_P	6,512,646.398	6,512,648.397	6,512,650.397
Initial ordinate X_P	6,015,861.876	6,015,861.827	6,015,861.778
Chainage of the *KP1* origin	36.896	28.284	20.972
Chainage of the *KP1* end	134.668	143.284	150.608
Chainage of the *ŁK* centre	794.345	792.642	790.944
Chainage of the *KP2* end	1454.021	1442.001	1431.280
Chainage of the *KP2* origin	1551.794	1557.001	1560.916
Chainage of the system end	1588.690	1585.285	1581.888
End abscissa x_K	1376.378	1374.964	1373.550
End ordinate y_K	18.586	20.000	21.414
End abscissa Y_K	6,513,658.658	6,513,658.609	6,513,658.561
End ordinate X_K	6,016,825.802	6,016,823.802	6,016,821.803

[Author's study].

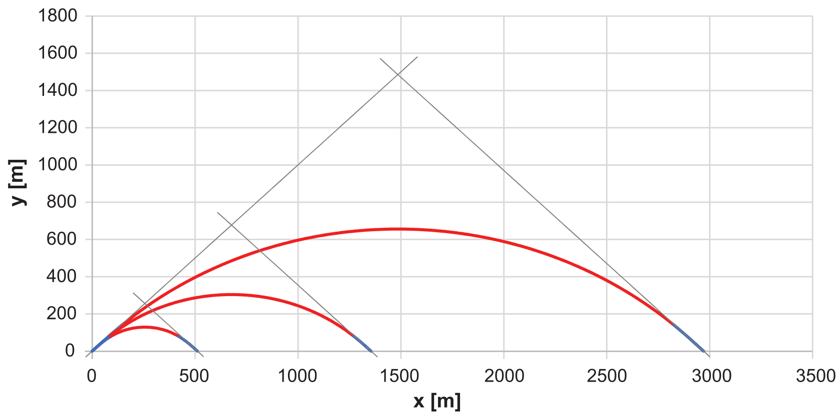


Fig. 8. Curve deflection areas from Table 4 for R_m = 300 m, 900 m and 2000 m (in order from left); circular curves are marked in red and transition curves in blue [author's study]

Table 4

A list of the calculated geometric parameters for the selected curve deflection areas (curve deflection angle $\alpha = \pi/2$ rad)

Geometric parameter	Case 1 V = 80 km/h, h = 130 mm	Case 2 V = 120 km/h, h = 80 mm	Case 3 V = 180 km/h, h = 100 mm
Intertrack space			
Curve radius R_m [m]	300	900	2,000
Intertrack space widening Δb_m [mm]	670	340	370
Length of transition curves l_m [m]	120	115	200
Abscissa of the circular curve midpoint x_{Sm} [m]	255.9141	677.4821	1,485.5076
Ordinate of the circular curve midpoint Y_{Sm} [m]	128.8257	303.8241	655.9020
Outer track			
Curve radius R_z [m]	302.335	902.170	2,002.185
Length of the transition curves l_z [m]	109.8855	97.8525	176.501
Abscissa of the circular curve midpoint x_{Sz} [m]	255.9141	677.4821	1,485.5076
Ordinate of the circular curve midpoint Y_{Sz} [m]	131.1607	305.9942	658.0870
Inner track			
Curve radius R_w [m]	297.665	897.830	1,997.815
Length of the transition curves l_w [m]	129.1851	129.8361	220.9714
Abscissa of the circular curve midpoint x_{Sw} [m]	255.9141	677.4821	1485.5076
Ordinate of the circular curve midpoint y_{Sw} [m]	126.4907	301.6541	653.7170
Verification of calculations			
Intertrack space width d_m [m]	4.6700	4.3401	4.3700

[Author's study].

space widening parameters can be applied to the full range of circular curve radii and meets all the requirements in terms of the required accuracy.

10. Conclusion

The work addresses the issue of designing the geometry of a curved double track using the analytical design method principles. This allowed, similarly to other applications of the method, a complete overview of the issue and the method for determining the key parameters to be defined. An analysis for an elementary (symmetrical) geometry composed of a circular curve and two transition curves of the same type and length has been assumed.

The analytical method for determining the intertrack space axis and the outer and inner track axes has been introduced, leading to the required value of the intertrack space widening. The widening itself is achieved by varying the length of the transition curves in the outer and inner tracks. Appropriate mathematical expressions have been formulated from which the lengths of the corresponding transition curves can be determined iteratively. With the track axis coordinates in the local coordinate system, these can be easily transferred to the PL-2000 two-dimensional Cartesian coordinate system, i.e., an element of the national spatial reference system.

The analysis continued with the issue of the chainage of axes of the intertrack space and the mainline tracks. In the calculation example, linear coordinate and Cartesian coordinate values have been determined for the individual characteristic points of the intertrack space axis (including hectometric posts) to allow easy determination of these points in the field and are useful for creating line chainage.

The applicability of the proposed method for determining the intertrack space widening parameters has been determined by a detailed analysis performed for several cases, characterised by significantly deviating values of circular curve radius. This analysis verified the high precision of the calculations performed and demonstrated that the method is applicable to the full range of radii of circular curves and meets all requirements in terms of the required accuracy.

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