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Basic Variants of the Analytical Method of Designing Track Geometric Layouts

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Summary

The article presents (and extends) the basic assumptions of the analytical method for designing track geometric layouts. The individual elements of the layout (straight sections, circular arcs and transition curves) are described using mathematical equations and connected with each other while maintaining the condition of tangent compliance. The method covers various design cases: a symmetrical case, with transition curves of the same type and the same length, an asymmetrical case, resulting from different types and lengths of transition curves, as well as methods of designing compound and reverse curves. The work presents a detailed design procedure for the typical, most common case in which the transition curves are symmetrical in relation to the circular arc. Two basic variants differing in the location of the local coordinate system are considered. In the standard (universal) variant, the location of the beginning of the system in the PL-2000 system is not known and is determined only in the final phase of the procedure. Due to this, some interpretation problems may arise. In the case of a symmetrical geometric layout, these difficulties can be avoided thanks to the introduced modification consisting of locating the origin of the local coordinate system at the intersection of two main directions of the route. The article presents computational algorithms for both discussed variants. The benefits of the introduced modification are illustrated by the presented computational examples.

Keywords: railroad, analytical design method, modification of the local coordinate system, calculation algorithm, sample geometric layouts

1. Introduction

For many years, commercial computer software has been used to develop design documentation in the field of railways [1, 2]. One might get the impression that this has had the effect of reducing the intensity of current research work on the methodology of designing track geometric layouts. However, such work is still being carried out [3–6] and its scope is often limited to specific issues such as transition curves [7] or railroad turnouts [8, 9]. In a recently published book [10], there is a small chapter devoted to the topic of designing track geometric layouts. However, various research works on determining the geometric layout of the track along the measurement path are being developed on a large scale [11–21].

In 2009, a research team of the Gdańsk University of Technology and the Naval Academy in Gdynia conducted field experiments involving the use of GNSS (*Global Navigation Satellite System*) receivers, installed on a moving rail vehicle, to determine the coordinates of railroad track axes [22]. Those activities, developed in the following years [23], were referred to as mobile satellite measurements.

Mobile satellite measurements provide a huge number of coordinates in a very short time – the currently used receivers have a frequency of up to 100 Hz. In 2009, this frequency was much lower, but still - compared to traditional geodetic measurements – the number of obtained measurement data was incomparably greater. This situation was the inspiration to adapt the methodology for designing track geometric layouts to satellite measurements.

The article [24] presents for the first time the assumptions of the so-called analytical design method (ADM); more precisely, concerning the design of the area of the route direction change. In this method, the individual elements of the layout (straight segments, circular arcs and transition curves) are described using mathematical formulas and connected with each other while maintaining the condition of tangent compliance. The paper [24] considers the

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symmetrical case characterized by transition curves of the same type and length. The work [25] provides a generalization of the method under study and concerns the design of an asymmetrical system created by varying the type and length of transition curves. Using an analogous approach to the given cases, a method for designing compound [26] and reverse [27] curves was also developed.

2. Basic assumptions of the analytical design method

The basis for designing a railroad route in the horizontal plane is the creation of its polygon, that is, a system of intersecting main directions. The route polygon is located on a site and elevation plan specified in the relevant state spatial reference system. In Poland – with regard to plane coordinates – the PL-2000 system [28] is in force, created based on a mathematically unambiguous assignment of points on the GRS 80 reference ellipsoid [29] to corresponding points on the plane according to the Gauss-Krüger projection [30]. The individual main directions can be written in the form of a formula:

$$X = A_i + B_i Y, \tag{1}$$

where Y is the eastern plane coordinate values, X – the northern coordinate values, while A_i and B_i are the coefficients of the equation of a given line.

From the point of view of determining the actual direction of the route, the key value is the slope coefficient of the straight line $B_i = \tan \varphi_i$. Determining the angles of inclination of the adjacent straight lines φ_i and φ_{i+1} with respect to the Y axis allows to determine the basic data for design – the turning angle α of the route. The values of φ_i and φ_{i+1} can be positive or negative, but the following conditions must be met: $\varphi_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\varphi_{i+1} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Depending on the geometric layout, angle α is determined using one of two formulas:

$$\alpha = \left|\varphi_{i+1} - \varphi_i\right| \tag{2a}$$

or

$$\alpha = \pi - \left|\varphi_{i+1} - \varphi_i\right| \tag{2b}$$

Knowledge of the equations of the adjacent main directions of the route makes it possible to determine the coordinates of their point of intersection, i.e. point $W(Y_W, X_W)$. The Y_W and X_W coordinates are as follows:

$$Y_W = \frac{A_{i+1} - A_i}{B_i - B_{i+1}} \tag{3}$$

$$X_W = A_i + \frac{A_{i+1} - A_i}{B_i - B_{i+1}} B_i \tag{4}$$

In the method under study, however, the design of the area of the route direction change is not done in the PL-2000 system, but in the corresponding local Cartesian x, y coordinate system, the basis of which is the symmetrically aligned adjacent main directions. Figure 1 shows two intersecting main directions (line i and line i+1), with the origin of the PL-2000 system shifted to the selected point $O(Y_O, X_O)$, located on line i. The presented case applies to the standard (i.e. universal) variant of the analytical design method. The detailed design procedure for that variant is presented in section 3 of this article.

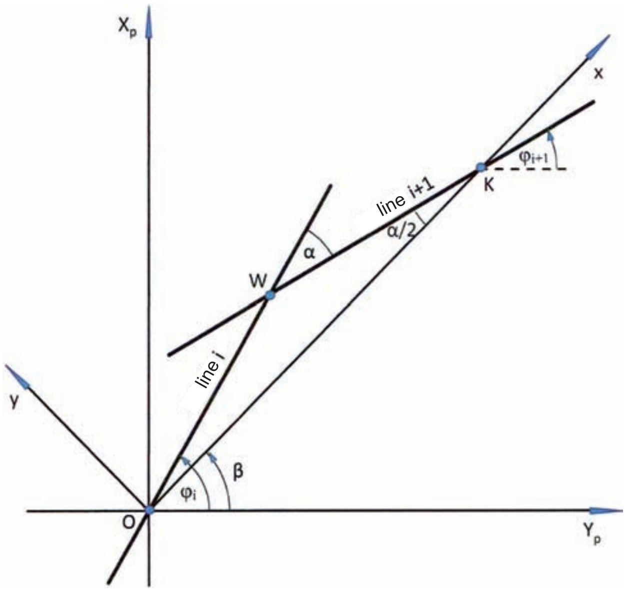

Fig. 1. Local x, y coordinate system in the ADM standard variant against the background of the intersecting directions of the main route, in the shifted PL-2000 system [own elaboration]

The next section describes a modified variant of the analytical method for designing track geometric layouts. In that variant, the general image of it is shown in Figure 5, the origin of the PL-2000 system is moved to a clearly defined point $W(Y_W, X_W)$.

To obtain a symmetrical arrangement of the main directions, the displaced PL-2000 system must be additionally rotated by an appropriate angle β in both variants.

3. Standard design option

3.1. General characteristics

The process of the creation of a local x, y coordinate system in the standard design variant is illustrated in Figure 1. The origin of that system, the point $O(0,0)$, corresponds to the origin of the PL-2000 system shifted to the point $O(Y_O, X_O)$. However, to properly set the x-axis, the Y_P, X_P system must be rotated by an angle β with respect to point O. The value of that angle is determined using one of the formulas (5a) ÷ (5f).

$$\beta = \varphi_i - \frac{\alpha}{2} \tag{5a}$$

$$\beta = \varphi_i + \frac{\alpha}{2} \tag{5b}$$

$$\beta = \pi + \left(\varphi_i - \frac{\alpha}{2} \right) \tag{5c}$$

$$\beta = \pi + \left(\varphi_i + \frac{\alpha}{2} \right) \tag{5d}$$

$$\beta = \left(\varphi_i - \frac{\alpha}{2} \right) - \pi \tag{5e}$$

$$\beta = \left(\varphi_i + \frac{\alpha}{2} \right) - \pi \tag{5f}$$

Table 1 summarises the characteristics of all geometric cases. Y_{PW} and X_{PW} are the coordinates of the W vertex (in the shifted PL-2000 system). The symbols (+) and (–) indicate positive and negative values, while (L) and (R) indicate the directions of the route turn (rotation of the system) to the left or right. The numbering of the applicable formulas for α and β angles is also given.

In the local coordinate system, the x and y values are determined using the formulas [31]:

$$x = \left(Y - Y_O \right) \cos\beta + \left(X - X_O \right) \sin\beta \tag{6}$$

$$x = -\left(Y - Y_O \right) \sin\beta + \left(X - X_O \right) \cos\beta \tag{7}$$

whereby the positive value of the angle β is taken into account in the case of rotation of the system to the

Table 1

Characteristics of possible cases of the railroad route direction change in the standard design variant

No.	Y_{PW} abscissa	X_{PW} ordinate	Railroad route direction change	φ_i angle	φ_{i+1} angle	α angle	β angle	System rotation	y ordinate
1	(+)	(+)	(R)	(+)	(+)	(2a)	(5a)	(L)	(+)
2	(+)	(+)	(R)	(+)	(–)	(2a)	(5a)	(L)	(+)
3	(+)	(+)	(L)	(+)	(–)	(2b)	(5b)	(L)	(–)
4	(+)	(+)	(L)	(+)	(+)	(2b)	(5b)	(L)	(–)
5	(–)	(+)	(R)	(–)	(+)	(2b)	(5c)	(L)	(+)
6	(–)	(+)	(R)	(–)	(–)	(2b)	(5c)	(L)	(+)
7	(–)	(+)	(L)	(–)	(–)	(2b)	(5d)	(L)	(–)
8	(–)	(+)	(L)	(–)	(+)	(2a)	(5d)	(L)	(–)
9	(–)	(–)	(R)	(+)	(+)	(2a)	(5e)	(R)	(–)
10	(–)	(–)	(R)	(+)	(–)	(2a)	(5e)	(R)	(–)
11	(–)	(–)	(L)	(+)	(–)	(2b)	(5f)	(R)	(–)
12	(–)	(–)	(L)	(+)	(+)	(2b)	(5f)	(R)	(–)
13	(+)	(–)	(R)	(–)	(+)	(2b)	(5a)	(R)	(+)
14	(+)	(–)	(R)	(–)	(–)	(2b)	(5a)	(R)	(+)
15	(+)	(–)	(L)	(–)	(–)	(2a)	(5b)	(R)	(–)
16	(+)	(–)	(L)	(–)	(+)	(2a)	(5b)	(R)	(–)

[Own elaboration].

left, and its negative value in the case of rotation to the right. The inclination angles of the straight lines determining the main directions are as follows:

- for positive values of the y ordinates (cases 1, 2, 5, 6, 13, 14 in Table 1)

$$\bar{\varphi}_i = \frac{\alpha}{2},\ \bar{\varphi}_{i+1} = -\frac{\alpha}{2} \tag{8}$$

- with negative values of the y ordinates (cases 3, 4, 7 ÷ 12, 15, 16 in Table 1)

$$\bar{\varphi}_i = -\frac{\alpha}{2},\ \bar{\varphi}_{i+1} = \frac{\alpha}{2} \tag{9}$$

To design the geometric layout of the area of the route direction change, it is necessary to determine its basic parameters related to the assumed speed of trains V. Determination of the coordinates of the track axis in the x, y system requires a prior determination – in addition to the turning angle α of the route – of the following data:

- radius R of the circular arc,
- cant h_0 on the curve,
- type and length of the adopted transition curves.

The standard procedure of design, presented further down the text, applies to the typical most common case in which the transition curves are symmetrical in relation to the circular arc [24]. Both in this and other cases (described in the works [25–27]), in the initial phase, the location of the origin of the local x, y coordinate system in relation to the PL-2000 system is not specified, therefore it is of a contractual nature. Full integration of the local coordinate system (*LCS*) and the PL-2000 system requires the performance of the design procedure in the *LCS* almost to the very end.

Geometric layout design consists of several stages. The case of turning the route to the right is considered here (Figure 2). When the route is turned to the left, negative values of the y ordinates are obtained; to use the algorithm described below, it is necessary to mirror these ordinates with respect to the abscissa axis and use values $\bar{y} = -y$.

3.2. Determination of the coordinates of the transition curve in the auxiliary x_k, y_k coordinate system

The process starts by drawing through the $O_{LUW}(0,0)$ point in the local x, y coordinate system a straight line imitating the main direction i, inclined at an angle $\alpha/2$ to the x axis. The formula is as follows:

$$y = \tan\frac{\alpha}{2}x \tag{10}$$

The line (10) is the abscissa axis of the auxiliary coordinate system x_k, y_k, associated with the transition curve.

As established in the work [25], the form of a correct transition curve results from the equation of its curvature $\kappa(l)$, where l is the position of a given point of the curve measured along its length. The determined transition curve is presented using the parametric equations $x_k(l)$ and $y_k(l)$, for $l \in \langle 0, l_k \rangle$, where l_k is the length of the transition curve. For example, for a transition curve with a linear curvature, i.e. clothoid, these equations in the x_k, y_k system in Figure 2 are as follows:

$$x_k(l) = l - \frac{1}{40R^2 l_k^2} l^5 + \frac{1}{3456 R^4 l_k^4} l^5 \tag{11}$$

$$y_k(l) = -\frac{1}{6Rl_k} l^3 + \frac{1}{336 R^3 l_k^3} l^7 - \frac{1}{42240 R^5 l_k^5} l^{11} \tag{12}$$

The further course of action requires knowing the coordinates of the end point of the transition curve $x_k(l_k)$ and $y_k(l_k)$, as well as the angle of inclination of the curve $\theta(l_k)$ at that point. For the clothoid, the coordinates $x_k(l_k)$ and $y_k(l_k)$ are determined using formulas (11) and (12), while the $\theta_k(l_k)$ angle is determined using the formula:

$$\theta_k(l_k) = -\frac{l_k}{2R} \tag{13}$$

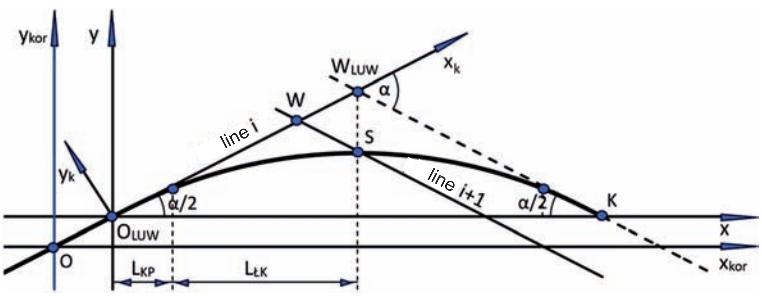

Fig. 2. Adopted local coordinate system in the ADM standard variant (symmetrical case) [own elaboration]

3.3. Transformation of the transition curve to the local *x*, *y* coordinate system

The next step is to transform the transition curve to the adopted local coordinate system by rotating its datum by the $\alpha/2$ angle. In the case under consideration, due to the clockwise direction of rotation, there are negative angle values in the applied transformation formulas [31].

$$x(l) = x_k(l)\cos\left(-\frac{\alpha}{2}\right) + y_k(l)\sin\left(-\frac{\alpha}{2}\right) \tag{14}$$

$$y(l) = -x_k(l)\sin\left(-\frac{\alpha}{2}\right) + y_k(l)\cos\left(-\frac{\alpha}{2}\right) \tag{15}$$

Since $\frac{\alpha}{2} \in \left\langle 0, \frac{\pi}{2} \right\rangle$, the following parametric equations of the transition curve in the local coordinate system are obtained:

$$x(l) = x_k(l)\cos\frac{\alpha}{2} - y_k(l)\sin\frac{\alpha}{2} \tag{16}$$

$$y(l) = x_k(l)\sin\frac{\alpha}{2} + y_k(l)\cos\frac{\alpha}{2} \tag{17}$$

The abscissa of the transition curve $x \in \left\langle 0, L_{KP} \right\rangle$, where:

$$L_{KP} = x(l_k) = x_k(l_k)\cos\frac{\alpha}{2} - y_k(l_k)\sin\frac{\alpha}{2} \tag{18}$$

The end ordinate of the transition curve is:

$$y_{KP} = y(l_k) = x_k(l_k)\sin\frac{\alpha}{2} + y_k(l_k)\cos\frac{\alpha}{2} \tag{19}$$

and the value of the tangent at the end

$$s_{KP} = \tan\left[\theta_k(l_k) + \frac{\alpha}{2}\right] \tag{20}$$

3.4. Determination of ordinates of circular arc

Knowing the position of the transition curve, a circular arc of radius R can be inserted into the geometric layout. The length of its projection on the x-axis, i.e. the value of $L_{ŁK}$, is determined based on tangency conditions:

- at the beginning of the arc, i.e. for $x = L_{KP}$, $y'(L_{KP}) = s_{KP}$,
- at the centre of the arc, i.e. for $x = L_{KP} + L_{ŁK}$, $y'(L_{KP} + L_{ŁK}) = 0$.

From these conditions, it follows that:

$$L_{LK} = \frac{s_{KP}}{\sqrt{1 + s_{KP}^2}} R \tag{21}$$

Now, the equation of the circular arc can be determined in the form of an explicit function $y = y(x)$. It has the following form:

$$\begin{gathered} y(x) = y_{KP} + \sqrt{R^2 - \left(L_{KP} + L_{LK} - x\right)^2} - \\ + \sqrt{R^2 - L_{LK}^2}, \\ x \in \left\langle L_{KP}, L_{KP} + L_{LK} \right\rangle \end{gathered} \tag{22}$$

The ordinate of the centre of the circular arc S is:

$$y(L_{KP} + L_{LK}) = y_{KP} + R - \sqrt{R^2 - L_{LK}^2} \tag{23}$$

3.5. Completion of coordinates for the second part of the designed route area

The solution presented so far covers half of the entire geometric layout, i.e. the region from the beginning of the transition curve to the centre of the circular arc. Thus, it is necessary to complete the ordinates for the second part of the designed area, i.e. for $x \in \left\langle L_{KP} + L_{LK}, 2\left(L_{KP} + L_{LK}\right) \right\rangle$. Due to symmetry, they will be a mirror image of the solution obtained for $x \in \left\langle 0, L_{KP} + L_{LK} \right\rangle$.

If the length of the projection of the entire system on the x-axis is denoted as L, where $L = 2(L_{KP} + L_{ŁK})$, then for the second transition curve, i.e. for $x \in \left\langle L - L_{KP}, L \right\rangle$, the abscissa equation is obtained:

$$\begin{gathered} x(l) = L - \left[x_k(l)\cos\frac{\alpha}{2} - y_k(l)\sin\frac{\alpha}{2} \right], \\ l \in \left\langle 0, l_k \right\rangle \end{gathered} \tag{24}$$

where $x_k(l)$ is defined by equation (11), and $y_k(l)$ – by equation (12). The equation of the ordinate $y(l)$ is described, as before, by formula (17).

For the other half of the circular arc, the following equation applies:

$$\begin{gathered} y(x) = y_{KP} + \sqrt{R^2 - \left(x - L_{KP} - L_{LK}\right)^2} - \\ + \sqrt{R^2 - L_{LK}^2} \\ x \in \left\langle L_{KP} + L_{LK}, L_{KP} + 2L_{LK} \right\rangle \end{gathered} \tag{25}$$

3.6. Determination of the position of the local coordinate system against the main directions of the route

Determining the coordinates of the centre of the circular arc S enables to determine the coordinates of the W_{LUW} point, which is the intersection of the main direction i and the line parallel to the main direction i+1 (Figure 2). The abscissa of the W_{LUW} point in the local coordinate system is the same as the abscissa of the S point, therefore it is equal to:

$$x_{W(LUW)} = x_S = L_{KP} + L_{LK} \quad (26)$$

Since the W_{LUW} point lies on the straight line imitating the main direction i, described by equation (10), the value of its ordinate is equal to

$$y_{W(LUW)} = \tan\frac{\alpha}{2} x_{W(LUW)} = \tan\frac{\alpha}{2}\left(L_{KP} + L_{LK}\right) \quad (27)$$

So far, the local coordinate system has been oriented with respect to one of the main directions (straight line i). By determining the coordinates of the W_{LUW} point, it is possible to consider the other direction as well (i.e. straight line i+1). In this way, the location of the LCS was clearly determined against the main directions of the route.

Knowing the coordinates of the W_{LUW} vertex makes it possible to determine its distance from the origin of the local coordinate system.

$$\overline{W_{LUW}O_{LUW}} = \sqrt{1+\left(\tan\frac{\alpha}{2}\right)^2}\left(L_{KP} + L_{LK}\right) \quad (28)$$

This allows the LCS position to be adjusted by taking into account the correct distance of its origin point from the W vertex (Figure 2). The origin of the adjusted coordinate system x_{kor}, y_{kor} is located at point O, the distance of which from the W vertex is $\overline{OW} = \overline{W_{LUW}O_{LUW}}$. This system is horizontally displaced relative to the x, y system by a value of

$$\Delta x = \overline{OW}\cos\frac{\alpha}{2} \quad (29)$$

and vertically by

$$\Delta y = \overline{OW}\sin\frac{\alpha}{2} \quad (30)$$

In the adjusted LCS, the coordinates of the designed geometric layout are the same as in the x, y system; which means that $x_{kor} = x$, $y_{kor} = y$.

3.7. Determination of coordinates of the point O(0,0) in the PL-2000 system

The distance between points O and W, defined by formula (28), can also be found in the PL-2000 system. There, the coordinates of point W are described by equations (3) and (4), while both points are found on line i. Thus, it is possible to determine the coordinates of point O in the PL-2000 system. The values of the coordinates are as follows:

$$Y_O = Y_W \pm \frac{1}{\sqrt{1+B_i^2}}\overline{OW} \quad (31)$$

$$Y_O = X_W \pm \frac{B_i}{\sqrt{1+B_i^2}}\overline{OW} \quad (32)$$

3.8. Transfer of the designed geometric layout to the PL-2000 system

Determining the Y_O and X_O coordinates using a predetermined rotation angle β makes it possible to transfer the obtained solution from the local coordinate system to the PL-2000 system. This is done using the following formulas [31]:

$$Y = Y_O + x\cos\beta - y\sin\beta \quad (33)$$

$$X = X_O + x\sin\beta - y\cos\beta \quad (34)$$

3.9. Calculation example I

In the PL-2000 system, the straight line representing the main direction i is described using the formula

$$X = 42{,}950{,}337.428 - 5.67128182\ Y,$$

and the straight line representing the direction i +1 – with the use of the formula:

$$X = 552{,}615.938 + 0.83909963\ Y$$

The coordinates of the intersection point of the main directions are: Y_W = 6,512,325.247 m, X_W = 6,017,105.651 m.

Due to the assumption of the location of point O on straight line i, with coordinates Y_O = 6,512,505.628 m and X_O = 6,016,082.661 m, in the shifted PL-2000 system, the W vertex has the following coordinates: Y_{PW} = −180.381 m and X_{PW} = 1022.990 m. Based on the given equations of the main directions, it follows that the inclination angles of the straight lines are: $\varphi_i = -1.396263$ rad and $\varphi_{i+1} = 0.698132$ rad. Since the route turns right, the given situation is case 5 in Table 1. The turning angle of the route – based on formula (2b) – is $\alpha = 1.047198$ rad, while the required turn angle - determined by formula (5c) – is $\beta = 1.221730$ rad. In the x, y coordinate system, the inclination angles of the straight lines will be: $\overline{\varphi}_i = 0.523599$ rad, $\overline{\varphi}_{i+1} = -0.523599$ rad (such an arrangement of inclinations is typical for the case when the route turns right).

The speed of trains on the designed geometric layout was assumed to be $V = 160$ km/h. This condition is met by a system with a radius $R = 1660$ m and a cant of $h_0 = 95$ mm, where the unbalanced acceleration $a_m = 0.569$ m/s^2. The required length of the transition curve in the form of a clothoid is $l_k = 160$ m (wheel lift speed on the gradient due to cant $f = 26.389$ mm/s).

The design procedure was initiated by determining the coordinates of the transition curve in the form of a clothoid in the x_k, y_k system. At its end point, the values $x_k(l_k) = 159.963$ m and $y_k(l_k) = -2.578$ m were obtained, as well as the tangent inclination angle: $\theta_k(l_k) = -0.04819$ rad. Rotation by $\alpha/2$ angle resulted in x, y coordinates of the curve. At its end, these amounted to: $x(l_k) = L_{KP} = 139.821$ m and $y(l_k) = 77.749$ m. At that point, the inclination angle of the tangent was $\theta(l_k) = 0.475406$ rad.

Using the formula (21), the value of $L_{ŁK} = 759.781$ m was determined. Knowledge of the L_{KP} and $L_{ŁK}$ made it possible to determine the coordinates of half of the circular arc using equation (22). The centre of the circular arc has the following abscissa: $x_S = L_{KP} + L_{ŁK} = 899.602$ m and an ordinate: $y_S = 261.831$ m.

The coordinates for the other half of the designed route area were then completed. Due to symmetry, they are a mirror image of the solution obtained for $x \in \langle 0, L_{KP} + L_{LK} \rangle$. The length of the projection of the whole system on the abscissa axis was 1,799.204 m. Figure 3 shows the designed geometric layout in the local coordinate system. Red indicates circular arc purple – transition curves, green – straight sections.

In the next step, the coordinates of the LCS origin point in the PL-2000 system were determined. The distance of that point from the W vertex was 1038.771 m. In the case under consideration, the coordinates of point O in the PL-2000 system were determined using the following formulas:

$$Y_O = Y_W + \frac{1}{\sqrt{1+B_i^2}}\overline{OW}$$

$$Y_O = X_W + \frac{B_i}{\sqrt{1+B_i^2}}\overline{OW}$$

Based on that $Y_O = 6{,}512{,}505.628$ m and $X_O = 6{,}016{,}082.661$ m were obtained. Figure 4 shows the designed geometric layout in the PL-2000 system.

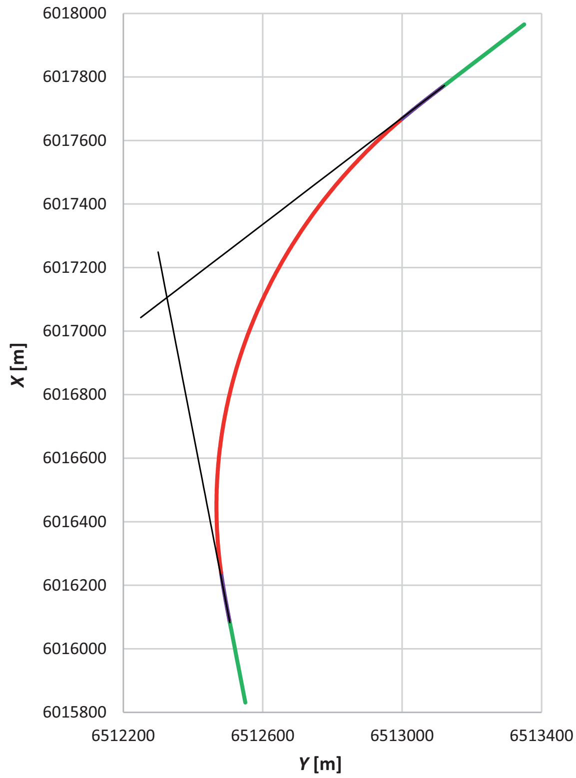

Fig. 4. Example I: geometric layout designed in the ADM standard variant in the PL-2000 system [own elaboration]

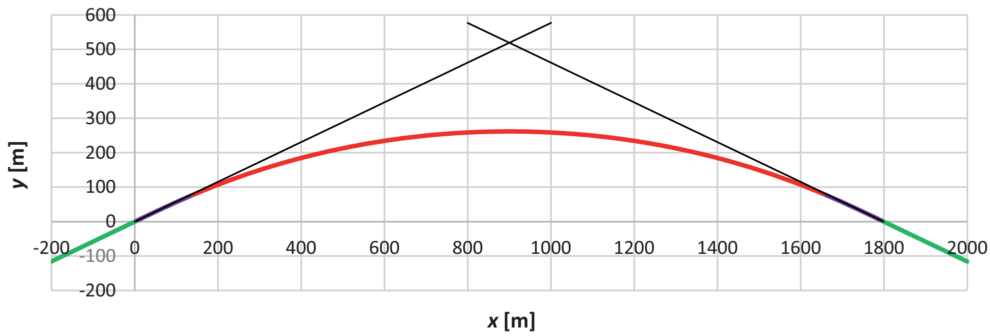

Fig. 3. Example I: geometric layout designed in the ADM standard variant in the local coordinate system [own elaboration]

4. Modified design variant

4.1. General characteristics

One methodological reservation could be made to the presented standard design method regarding the operation of an ambiguously defined local coordinate system during the implementation of the relevant procedure. The position of the origin point of the system with respect to the corresponding main point of the route and the resulting coordinates in the PL-2000 system are determined only in the final phase. Some interpretation problems may arise, which result from the need to correctly select the symbols in formulas (31) and (32). Moreover, Y_O and X_O coordinates play a key role in transferring the obtained solution from the local coordinate system to the PL-2000 system, using formulas (33) and (34).

In the case of a symmetrical geometric layout, those difficulties can be avoided by not using the $O(Y_O, X_O)$ point. This is possible by locating the origin of the local coordinate system at the intersection of the two main directions, whose Y_W and X_W coordinates are known. The way of creation of a modified local coordinate system is shown in Figure 5.

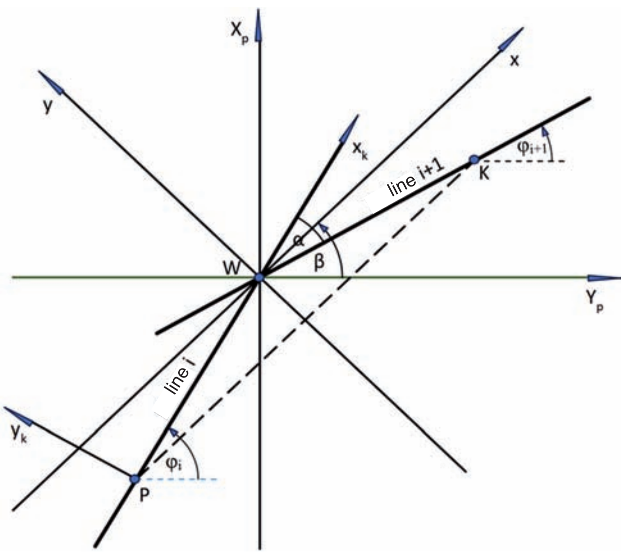

Fig. 5. Local *x*, *y* coordinate system in the ADM modified variant against the background of the intersecting directions of the main route, in the shifted PL-2000 system [own elaboration]

As before, to obtain a symmetrical arrangement of the main directions, it is necessary to make the appropriate transformation (shift and rotation) of the PL-2000 system. However, this time the origin of the PL-2000 system is shifted to the $W(Y_W, X_W)$ point, while the rotation is made in relation to that point by β angle until a symmetrical alignment of the two straight lines is achieved. The value of the angle β is determined using one of the formulas (35a) ÷ (35g).

$$\beta = \varphi_{i+1} + \frac{\alpha}{2} \tag{35a}$$

$$\beta = \varphi_{i+1} - \frac{\alpha}{2} \tag{35b}$$

$$\beta = \pi - \left(\varphi_{i+1} + \frac{\alpha}{2} \right) \tag{35c}$$

$$\beta = \pi - \left(\varphi_{i+1} - \frac{\alpha}{2} \right) \tag{35d}$$

$$\beta = \pi + \left(\varphi_{i+1} + \frac{\alpha}{2} \right) \tag{35e}$$

$$\beta = \pi + \left(\varphi_{i+1} - \frac{\alpha}{2} \right) \tag{35f}$$

$$\beta = \left(\varphi_{i+1} + \frac{\alpha}{2} \right) - \pi \tag{35g}$$

Table 2 summarises the characteristics of all geometric cases. Y_{PP} and X_{PP} are the coordinates of the origin in the shifted PL-2000 system. The symbols (+) and (–) indicate positive and negative values, while (L) and (R) indicate the directions of the route turn to the left or right. The numbering of the applicable formulas for α and β angles is also given. There are several stages of the design process of the geometric layout, which are outlined in sections 4.2 ÷ 4.4.

4.2. Determination of basic characteristic quantities

To be able to operate in the local coordinate system, the basic characteristic quantities must be determined. Those are the lengths of the projections of the transition curve (L_{KP}) and half of the circular arc ($L_{ŁK}$) on the horizontal axis, as well as the lengths of the projections of the transition curve (Δy_{KP}) and the centre of the circular arc ($\Delta y_{ŁK}$) on the vertical axis. There is – to a limited extent – an analogy to the course of the procedure presented in sections 3.2 ÷ 3.4. The calculation of the required parameters is carried out in the Δx, Δy system shown in Figure 6. In the given

Table 2

Characteristics of possible cases of the railroad route direction change in the modified design variant

No.	Y_{PP} abscissa	X_{PP} ordinate	Railroad route direction change	φ_i angle	φ_{i+1} angle	Difference $\varphi_{i+1} - \varphi_i$	α angle	β angle
1	(–)	(–)	(R)	(+)	(+)	(–)	(2a)	(35a)
2	(–)	(–)	(R)	(+)	(–)	(–)	(2a)	(35a)
3	(–)	(–)	(R)	(+)	(+)	(+)	(2b)	(35g)
4	(–)	(–)	(L)	(+)	(+)	(+)	(2a)	(35b)
5	(–)	(–)	(L)	(+)	(–)	(–)	(2b)	(35f)
6	(–)	(–)	(L)	(+)	(+)	(–)	(2b)	(35f)
7	(+)	(–)	(R)	(–)	(–)	(–)	(2a)	(35c)
8	(+)	(–)	(R)	(–)	(+)	(–)	(2b)	(35a)
9	(+)	(–)	(R)	(–)	(–)	(–)	(2b)	(35a)
10	(+)	(–)	(L)	(–)	(–)	(+)	(2a)	(35f)
11	(+)	(–)	(L)	(–)	(+)	(+)	(2a)	(35c)
12	(+)	(–)	(L)	(–)	(–)	(–)	(2b)	(35b)
13	(+)	(+)	(R)	(+)	(+)	(+)	(2b)	(35a)
14	(+)	(+)	(R)	(+)	(–)	(–)	(2b)	(35e)
15	(+)	(+)	(R)	(+)	(+)	(–)	(2a)	(35e)
16	(+)	(+)	(L)	(+)	(+)	(+)	(2a)	(35d)
17	(+)	(+)	(L)	(+)	(–)	(–)	(2b)	(35b)
18	(+)	(+)	(L)	(+)	(+)	(–)	(2b)	(35b)
19	(–)	(+)	(R)	(–)	(–)	(–)	(2a)	(35a)
20	(–)	(+)	(R)	(–)	(+)	(+)	(2b)	(35g)
21	(–)	(+)	(R)	(–)	(–)	(+)	(2b)	(35g)
22	(–)	(+)	(L)	(–)	(–)	(+)	(2a)	(35b)
23	(–)	(+)	(L)	(–)	(+)	(+)	(2a)	(35b)
24	(–)	(+)	(L)	(–)	(–)	(–)	(2b)	(35f)

[Own elaboration].

case, it refers to the turn of the route to the right, with a positive value of the inclination angle of the main route i in the x, y system. In the case of a leftward turn of the route, when the values of the inclination angle of the main route i in *LCS* are negative, it is enough to direct the Δy axis downwards (and adjust the y_k axis accordingly) to obtain a full analogy to the case of a rightward turn of the route considered below.

The design process was initiated by drawing through the $O(0, 0)$ point in the Δx, Δy coordinate system a straight line imitating the main direction i, described with the formula:

$$\Delta y = \tan\frac{\alpha}{2}\Delta x \tag{36}$$

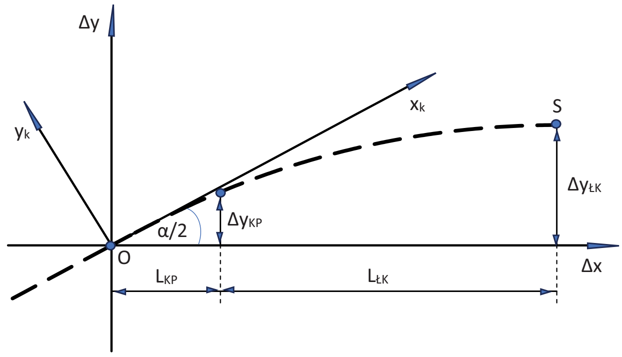

Fig. 6. The way of determining the basic characteristic quantities of a geometric layout [own elaboration]

The straight line is the abscissa axis of the auxiliary coordinate system x_k, y_k, associated with the transition curve. What matters are the coordinates of the end point of the curve in this system, which are derived from the corresponding parametric equations $x(l)$ and $y(l)$ for $l = l_k$. For the clothoid, those coordinates are as follows:

$$x_k\left(l_k\right) = l - \frac{l_k^3}{40R^2} + \frac{l_k^5}{3456R^4} \tag{37}$$

$$y_k\left(l_k\right) = -\frac{l_k^2}{6R} + \frac{l_k^4}{336R^3} - \frac{l_k^6}{42240R^5} \tag{38}$$

while the inclination angle of the $\theta_k(l_k)$ curve is determined using the formula (13).

The transformation of the transition curve to the Δx, Δy coordinate system is done by rotating its datum by the $\alpha/2$ angle. As a result of that operation, the required value of the projection of the transition curve on the horizontal axis:

$$L_{KP} = \Delta x\left(l_k\right) = x_k\left(l_k\right)\cos\frac{\alpha}{2} - y_k\left(l_k\right)\sin\frac{\alpha}{2} \tag{39}$$

and vertical axis

$$\Delta y_{KP} = \Delta y\left(l_k\right) = x_k\left(l_k\right)\sin\frac{\alpha}{2} + y_k\left(l_k\right)\cos\frac{\alpha}{2} \tag{40}$$

is obtained.

The value of the tangent at the end is determined using the formula (20). For a given radius R of a circular arc, the length of the projection of $L_{ŁK}$ on the horizontal axis can be directly determined using the formula (21). The value of the projection of the centre of the circular arc on the vertical axis is as follows:

$$\Delta y_{ŁK} = \Delta y\left(L_{KP} + L_{LK}\right) = \Delta y_{KP} + R - \sqrt{R^2 - L_{LK}^2} \tag{41}$$

It should also be taken into account that depending on the slope of the straight line i (positive or negative), the position of the designed geometric layout in the local coordinate system will be different. Therefore, a separate calculation procedure must be used for both cases.

There are several stages of the design of the geometric layout, starting with the assumption of an auxiliary x_k, y_k coordinate system associated with the transition curve. The abscissa axis of this system is located on the main direction i, and its origin (point P) coincides with the origin of the designed geometric layout. The course of the procedure involves two cases, determined by the location of this arrangement relative to the W vertex.

4.3. Determination of coordinates in the *x*, *y* system for a geometric layout located below the vertex *W*

For a geometric layout located below the vertex W, there is a positive inclination of the main direction i (Figure 7).

Knowing the lengths of the projections of the transition curve (L_{KP} and Δy_{KP}) and half of the circular arc ($L_{ŁK}$ and $\Delta y_{ŁK}$) on the axes of the coordinate system allows one to create a modified local coordinate system with marked characteristic points. The layout points that occur are as follows: W – intersection of the main directions of the route (beginning of the LCS), P – beginning of the geometric layout, K_1 – end of the first transition curve, S – centre of the circular arc, K_2 – end of the second transition curve, K – end of the geometric layout. The coordinates of the origin point P are:

$$x_P = -\left(L_{KP} + L_{LK}\right) \tag{42}$$

$$y_P = x_P \tan\frac{\alpha}{2} = -\tan\frac{\alpha}{2}\left(L_{KP} + L_{LK}\right) \tag{43}$$

4.3.1. First transition curve (for $x \in \left\langle -\left(L_{KP} + L_{LK}\right), -L_{LK} \right\rangle$)

As stated in section 3.2, the transition curve is written using the parametric equations $x_k(l)$ and $y_k(l)$, for $l \in \left\langle 0, l_k \right\rangle$. For the clothoid, the coordinates $x_k(l)$ and $y_k(l)$ are determined using formulas (11) and (12),

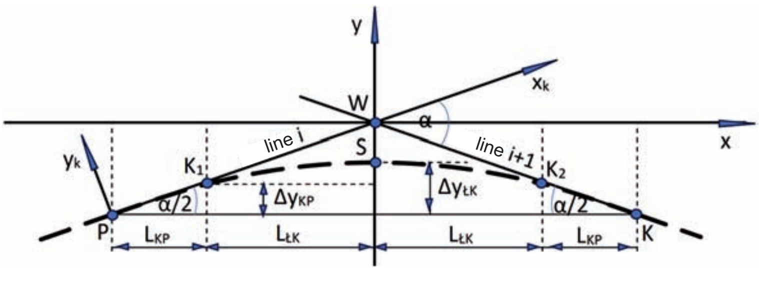

Fig. 7. Adopted local coordinate system in the modified AMP variant with marked characteristic points for the case of a geometric layout located below the vertex W [own elaboration]

while angle of inclination of the tangent $\theta_k(l_k)$ is determined using the formula (13).

The next step is to transform the transition curve to the adopted local coordinate system. In the case at hand, the transformation consists of rotating the x_k, y_k system to the right by an angle $\alpha/2$ and taking into account the location of point P in the LCS. The following parametric equations of the transition curve are obtained:

$$x(l) = -(L_{KP} + L_{LK}) + x_k(l)\cos\frac{\alpha}{2} - y_k(l)\sin\frac{\alpha}{2} \tag{44}$$

$$y(l) = -\tan\frac{\alpha}{2}(L_{KP} + L_{LK}) + x_k(l)\sin\frac{\alpha}{2} + y_k(l)\cos\frac{\alpha}{2} \tag{45}$$

The abscissa of the end point of the transition curve (i.e. K_1 point) is $x_{K1} = -L_{ŁK}$, while the end ordinate of the curve results from the formula:

$$y_{K1} = -\tan\frac{\alpha}{2}(L_{KP} + L_{LK}) + x_k(l_k)\sin\frac{\alpha}{2} + y_k(l_k)\cos\frac{\alpha}{2} \tag{46}$$

The value of the tangent at the end is determined using the formula (20).

4.3.2. Circular arc (for $x \in \langle -L_{LK}, L_{LK} \rangle$)

The equation of the entire circular arc can be written in the form of an explicit function $y = y(x)$; it has the following form:

$$y(x) = y_{K1} + \sqrt{R^2 - x^2} - \sqrt{R^2 - L_{LK}^2} \tag{47}$$

The ordinate of the centre of the circular arc (i.e. S point) is:

$$y_S = y(0) = y_{K1} + R - \sqrt{R^2 - L_{LK}^2} \tag{48}$$

4.3.3. Second transition curve (for $x \in \langle L_{LK}, L_{KP} + L_{LK} \rangle$)

Due to the symmetry of the designed geometric layout, in the parametric equations of the second transition curve, the ordinate values of $y(l)$ are described – as before – by formula (45). However, the equation of the abscissa is different, taking the following form:

$$x(l) = L_{KP} + L_{LK} - \left[x_k(l)\cos\frac{\alpha}{2} - y_k(l)\sin\frac{\alpha}{2} \right] \tag{49}$$

4.3.4. Transfer of the designed geometric layout to the PL-2000 system

The Y_W and X_W coordinates, known from the outset, as well as the previously determined rotation angle β, make it possible to transfer the obtained solution from the local coordinate system to the PL-2000 system. This is done using the following formulas:

$$Y = Y_W + x\cos\beta - y\sin\beta \tag{50}$$

$$X = X_W + x\sin\beta + y\cos\beta \tag{51}$$

4.4. Determination of coordinates in the *x*, *y* system for a geometric layout located above the vertex *W*

For a geometric layout located above the vertex W, there is a negative inclination of the main direction i (Figure 8).

The abscissa of the origin point P follows – as before – from formula (42), while the ordinate – from the relationship:

$$y_P = \tan\frac{\alpha}{2}(L_{KP} + L_{LK}) \tag{52}$$

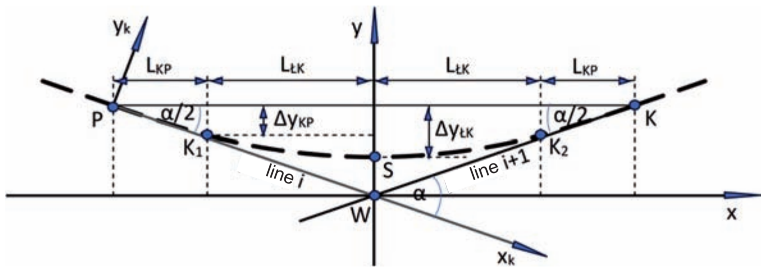

Fig. 8. Adopted local coordinate system in the modified ADM variant with marked characteristic points for the case of a geometric layout located above the vertex W [own elaboration]

4.4.1. First transition curve (for $x \in \left\langle -\left(L_{KP}+L_{LK}\right), -L_{LK} \right\rangle$)

First transition curve is written using the parametric equations $x_k(l)$ and $y_k(l)$, for $l \in \left\langle 0, l_k \right\rangle$. In the x_k, y_k coordinate system in Figure 8, the equation $x(l)$ takes – as before – the form of (11), while $y_k(l)$ is as follows:

$$y_k\left(l\right) = \frac{1}{6Rl_k} l^3 - \frac{1}{336R^3 l_k^3} l^7 + \frac{1}{42240R^5 l_k^5} l^{11} \quad (53)$$

The angle of inclination of the tangent $\theta_k(l_k)$ is described by the formula:

$$\theta_k\left(l_k\right) = \frac{l_k}{2R} \quad (54)$$

The transformation of the transition curve to the adopted local coordinate system consists, in the given case, of rotating the x_k, y_k system to the left by an angle $\alpha/2$. The following parametric equations of the transition curve in the *LCS* are obtained:

$$x\left(l\right) = -\left(L_{KP}+L_{LK}\right) + x_k\left(l\right)\cos\frac{\alpha}{2} + y_k\left(l\right)\sin\frac{\alpha}{2} \quad (55)$$

$$y\left(l\right) = \tan\frac{\alpha}{2}\left(L_{KP}+L_{LK}\right) - x_k\left(l\right)\sin\frac{\alpha}{2} + y_k\left(l\right)\cos\frac{\alpha}{2} \quad (56)$$

The abscissa of the end point of the transition curve (i.e. K_1 point) is $x_{K1} = -L_{ŁK}$, while the end ordinate of the curve results from the formula:

$$y_{K1} = \tan\frac{\alpha}{2}\left(L_{KP}+L_{LK}\right) - x_k\left(l_k\right)\sin\frac{\alpha}{2} + y_k\left(l_k\right)\cos\frac{\alpha}{2} \quad (57)$$

The value of the tangent at the end is determined using the formula:

$$s_{KP} = \tan\left[\theta_k\left(l_k\right) - \frac{\alpha}{2}\right] \quad (58)$$

4.4.2. Circular arc (for $x \in \left\langle -L_{LK}, L_{LK} \right\rangle$)

The equation of the entire circular arc is as follows:

$$y\left(x\right) = y_{K1} - \sqrt{R^2 - x^2} + \sqrt{R^2 - L_{LK}^2} \quad (59)$$

The ordinate of the centre of the circular arc (i.e. *S* point) is:

$$y_S = y\left(0\right) = y_{K1} - R + \sqrt{R^2 - L_{LK}^2} \quad (60)$$

4.4.3. Second transition curve (for $x \in \left\langle L_{LK}, L_{KP}+L_{LK} \right\rangle$)

Due to the symmetry of the designed geometric layout, in the parametric equations of the second transition curve, the ordinate values of $y(l)$ are described – as before – by formula (56). However, the equation of the abscissa is different, taking the following form:

$$x\left(l\right) = L_{KP} + L_{LK} - \left[x_k\left(l\right)\cos\frac{\alpha}{2} + y_k\left(l\right)\sin\frac{\alpha}{2}\right] \quad (61)$$

As in section 4.3.4, the transfer of the designed geometric layout to the PL-2000 system is done using formulas (50) and (51).

4.5. Calculation example II

In the PL-2000 system, the straight line representing the main direction i is described by the formula:

$$X = 17{,}900{,}192.150 - 1.73205081\ Y$$

and the straight line representing the direction $i+1$ – with the use of the formula:

$$X = 8{,}696{,}475.885 - 0.36397023\ Y$$

The coordinates of the intersection point of the main directions are: Y_W = 6,727,466.528 m, X_W = 6,247,878.317 m.

Due to the symmetry of the designed geometric layout, the origin of the local coordinate system was located at the intersection of the two main directions. Thus, in the shifted PL-2000 system, it is the $W(0, 0)$ point. Based on the given equations of the main directions, it follows that the inclination angles of the straight lines are: $\varphi_i = -1.047198$ rad and $\varphi_{i+1} = -0.349066$ rad. Since the route turns left here, the given situation is covered by case 22 in Table 2.

The curve deflection angle – based on formula (2a) – is $\alpha = 0.698132$ rad, while the required angle of rotation – determined by formula (35b) – is equal to $\beta = -0.698132$ rad (meaning a clockwise rotation). In the x, y coordinate system, the inclination angles of the straight lines will be: $\overline{\varphi}_i = -0.349066$ rad, $\overline{\varphi}_{i+1} = 0.349066$ rad (such an arrangement of inclinations is a typical situation for the case of a left turn of the route, which occurs with a negative inclination of the main direction i in the x, y system).

The speed of trains on the designed geometric layout was assumed to be V = 120 km/h. This condition is met by a layout with a radius R = 850 m and a cant of h_0 = 110 mm, where the unbalanced acceleration a_m = 0.588 m/s^2. The required length of the transition curve in the form of a clothoid is l_k = 135 m (wheel lift speed on the gradient due to cant f = 27.160 mm/s).

The first step was to perform an auxiliary operation to determine the values of L_{KP} and Δy_{KP} as well as $L_{ŁK}$ and $\Delta y_{ŁK}$. In the given case, the main direction i has a negative inclination in the x, y system. Therefore – unlike the situation in Figure 6 – the Δy axis must be directed downward, and the y_k axis adjusted accordingly. It was first necessary to determine the coordinates of the end of the transition curve in the x_k, y_k coordinate system using formulas (37) and (38). They are: $x_k(l_k)$ = 134.915 m and $y_k(l_k)$ = –3.609 m. The angle of inclination of the tangent is equal to $\theta_k(l_k)$ = –0.079412 rad. Rotation by an angle $\alpha/2$ to the left yielded – using formulas (39) and (40) – the sought-after values of L_{KP} and Δy_{KP}; they are L_{KP} = 128.000 m and Δy_{KP} = 42.787 m. The $\theta(l_k)$ angle necessary to determine $L_{ŁK}$, based on formula (20), is equal to $\theta(l_k)$ = 0.26965 rad. In this situation, the values of $L_{ŁK}$ and $\Delta y_{ŁK}$ can be determined using formulas (21) and (41); they are equal to $L_{ŁK}$ = 226.438 m and $\Delta y_{ŁK}$ = 73.504 m.

Further design operations are performed in the local x, y coordinate system. An auxiliary x_k, y_k coordinate system associated with the first transition curve is assumed. The beginning of this curve is also the beginning of the designed geometric layout, and its coordinates in the *LCS* are: x_P = –354.439 m, y_P = 129.005 m.

Clothoid coordinates $x_k(l)$ and $y_k(l)$ were determined using formulas (11) and (53) for $l \in \langle 0;135 \rangle$ m. The value of the tangent inclination at the end of the curve – based on formula (54) – was $\theta_k(l_k)$ = 0.079412 rad. The next step is to transform the transition curve to the adopted local coordinate system. In the case at hand, the transformation consists of rotating the x_k, y_k system to the left by an angle $\alpha/2$ and taking into account the location of point P in the *LCS*. The parametric equations $x(l)$ and $y(l)$ follow from formulas (55) and (56), and the condition $x \in \langle -354,439;-226,426 \rangle$ m applies. The abscissa of the end point of the transition curve (i.e. K_1 point) is x_{K1} = –226.426 m, while the end ordinate of the curve y_{K1} = 86.249 m. The angle of inclination of the tangent is equal to $\theta(l_k)$ = –0.26965 rad.

Knowledge of the L_{KP} and $L_{ŁK}$ made it possible to determine the ordinate of the circular centre (for x_S = 0) using equation (60). It is y_S = 55.536 m. The coordinates for the other half of the designed route area were then completed. Due to symmetry, they are a mirror image of the solution obtained for $x \in \langle 0, L_{KP} + L_{ŁK} \rangle$. The length of the projection of the whole layout on the abscissa axis was 708.877 m. Figure 9 shows the designed geometric layout in the local coordinate system. Red indicates circular arcs, purple – transition curves, green – straight sections.

In the next step, the obtained solution was transformed to the PL-2000 system, using formulas (50) and (51). The designed geometric layout is shown in Figure 10 (the colours of the indications are the same as in (Figure 9).

The presented calculation example shows the clear advantages that can be obtained by locating the origin of the local coordinate system at the point of intersection of the two main directions of the route (whose coordinates are known). A suitable calculation algorithm leads to the solution of the problem in a sequential manner, much more convenient than was previously the case.

5. Conclusions

The article presents (and extends) the basic assumptions of the analytical method for designing track geometric layouts. The design method was inspired by mobile satellite measurements initiated in 2009 by a research team from the Gdańsk University of Technology and the Naval Academy in Gdynia. These measurements provide a vast number of track axis coordinates in a very short time – the receivers

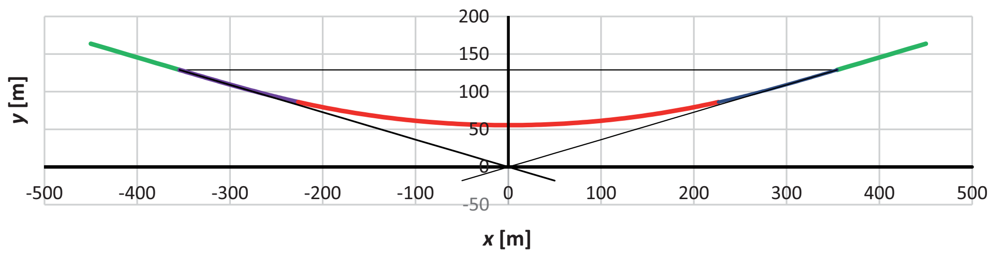

Fig. 9. Example II: geometric layout designed by modified analytical method in local coordinate system [own elaboration]

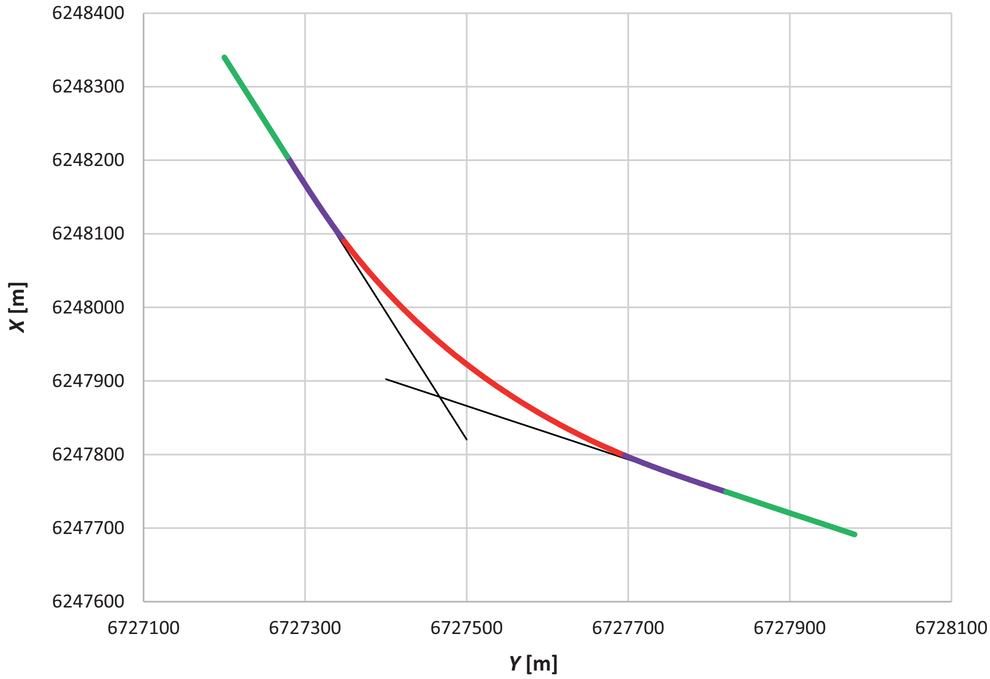

Fig. 10. Example II: geometric layout designed by modified analytical method in the PL-2000 system [own elaboration]

used today have frequencies of up to 100 Hz. This had to be utilised in some way, most simply by adapting the design method accordingly.

In the analytical design method, the individual elements of the geometric layout (straight sections, circular arcs and transition curves) are described using mathematical equations. The design of the area of the route direction change is carried out in the corresponding local Cartesian coordinate system, the basis of which is the symmetrically aligned adjacent main directions of the route. In order to be able to obtain such an arrangement of the main directions, it is necessary to make the appropriate transformation (shift and rotation) of the PL-2000 system. Designing a geometric layout requires first determining its basic parameters related to the assumed speed of trains.

The standard procedure of design presented in the article is characterized at its initial stage by the lack of knowledge of the position of the local coordinate system origin relative to the PL-2000 system. Full integration of these systems requires the design procedure to be carried out in the local system until the very end. The position of the origin point of the system with respect to the corresponding main point of the route and the resulting coordinates in the PL-2000 system are determined only in the final phase of the procedure. This may be the primary methodological reservation to the design method in question. Due to this, some interpretation problems may arise.

In the case of a symmetrical geometric layout, those difficulties can be avoided by setting the origin of the local coordinate system at the intersection of the two main directions of the route, the coordinates of which are known. Once the basic characteristic values are determined, the computational algorithm described in the article leads to solving the problem in a sequential way, much more convenient than before. The benefits are clearly illustrated by the two provided calculation examples.

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