

Modification of Reverse Curves on Railway Lines Using Connecting Curves

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Summary

This paper investigates methods for modifying reverse curves, i.e., a geometric layout consisting of two consecutive circular curves (usually with different radii), curving in opposite directions and directly connected to each other. These curves pose certain operational problems due to abrupt changes in curvature. The distance between the intersection points of adjacent main directions of the route, which cannot be too small, was discussed. On modernized railway lines, this requirement is frequently not satisfied, making it necessary to modify reverse curves accordingly. Three geometrical configurations of reverse arcs were considered. In Layout I, the distance between the main points of the route was significant (i.e., 1000 m). It was demonstrated that in this case, the standard solution involving the use of circular curves with transition curves could be successfully applied. In Layout II, a limited distance between the main points of the route was assumed (i.e., 400 m). From the perspective of train speeds, the standard solution did not meet the requirements in this case. Therefore, a curve connecting the two extreme main directions was proposed; this curve satisfies the appropriate boundary conditions and the assumed differential equation. By increasing the length of this curve, higher maximum train speeds can be achieved. One limitation may be the obstacles that arise, for example, due to the terrain conditions. Layout III is characterized by a short distance between the main points of the route (i.e., 200 m). The standard solution also proves unacceptable here for structural reasons, i.e., the incorporation of rolling stock wheels into the track with very small radii of the circular curves. Using an extended connecting curve may prove to be the only advantageous solution in this situation. The proposed method of modifying reverse curves provides a practical solution to design problems.

Keywords: railway lines, reverse curves, modification methods, extended connecting curve concept, train speed analysis

1. Introduction

Currently, high-speed rail is the focus of the most attention in rail transport [3, 4, 6, 21, 32]. Numerous studies also focus on rolling stock and its maintenance [23, 31, 35], and railway track-related research is being conducted on a large scale to examine the interactions between rail vehicles and the track [5, 7, 22, 29, 30]. At the same time, the view has become established that research into the design of track geometric layouts is now of somewhat lesser importance. This is undoubtedly connected with the widespread use of commercial computer software [18, 20] in the preparation of design documentation. Although such research is underway [1, 2, 8, 33], its scope is often limited to specific topics, such as transition curves [11, 12, 14, 34] or rail turnouts [9, 13, 19, 26].

New railway lines are generally designed to accommodate higher train operating speeds, and a substantial proportion of them are high-speed railways. Nevertheless, conventional railway lines – most of which were constructed in the nineteenth century and remain in operation today – continue to account for the vast majority of the railway network. There is little doubt that these lines should be upgraded to meet contemporary operational requirements. This applies particularly to railway lines traversing difficult terrain (e.g., mountainous regions), where horizontal curves of small radius occur, often together with geometrically controversial track layouts such as compound curves and reverse curves. Improving the performance of these lines and increasing train operating speeds require appropriate modernization measures. However, these lines appear to have attracted relatively little attention in recent research.

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In this context, it is important to emphasize the specific nature of the modernization process. Particular attention should be paid to the constraints imposed by the surrounding environment, which must be considered from a broader perspective. Railway lines incorporating compound and reverse curves have existed since the earliest days of railway development and have become integrated into long-established built-up areas. This existing development, together with the terrain, watercourses, and engineering structures, may significantly restrict the scope for alignment modification. Consequently, any new track alignment should accommodate the existing conditions to the greatest extent possible. At the same time, it must be recognized that, in some cases, technical and economic constraints may preclude a satisfactory solution.

This paper investigates methods for modifying reverse curves, i.e., a geometric layout consisting of two consecutive circular curves (usually with different radii), curving in opposite directions and directly connected to each other. These curves pose certain operational problems due to abrupt changes in curvature. They have also benefited to a lesser extent from advances in computational techniques introduced during the second half of the twentieth century. From a research perspective, the subject has attracted relatively limited attention, with the emphasis being placed primarily on its practical aspects [28].

A new approach to the design of reverse curves was presented in [16]. The study in question presents conventional solutions involving the creation of a geometric layout consisting of two circular arcs oriented in opposite directions and four transition curves connecting these curves to the corresponding main directions of the route. However, the study [16] also highlighted the limitations of the proposed method, emphasizing that reverse curves cannot be designed arbitrarily but must satisfy a range of geometric and site-specific constraints. Most importantly, an adequate distance must be maintained between the intersection points of adjacent main route directions (it must not be too short). On modernized railway lines, this requirement is frequently not satisfied, making it necessary to modify reverse curves accordingly.

As in other variants of the analytical design method, the computational procedure is based on a local coordinate system. The origin of this system is located at the intersection point of two adjacent main directions of the railway route, whose Cartesian coordinates are defined in the global system (in Poland, this is the national spatial reference system [25], within which the PL-2000 plane rectangular coordinate system is used). This implementation of the method was presented in [17]; it is universal and covers areas where main directions are connected in both sym-

metrical and asymmetrical arrangements. The same approach was subsequently applied to the design of reverse curves in [16].

Reverse curves, however, differ fundamentally from most applications of the analytical design method. In geometric layouts, two adjacent main directions of the route are most often identified and subjected to further processing (including the creation of a local coordinate system). In the case of reverse curves, the presence of a third main direction considerably complicates the problem and limits the generality of the proposed method. Nevertheless, the same principle is adopted for establishing the local coordinate system: it is defined using two selected adjacent main directions, which are arranged symmetrically with respect to the vertical coordinate axis.

To some extent, this approach follows the method for modeling compound curves on railway lines presented in [15]. However, the third main direction present in a reverse curve does not allow, as in the case of compound curves, a universal design method to be presented in symbolic form. Consequently, in [16], the successive stages of the reverse-curve design procedure were presented by applying the relevant theoretical relationships directly to a specific calculation example. At the same time, it was pointed out that the scope of application of the developed method could be expanded by analogy.

2. The Main Directions of the Route under Consideration

When considering the horizontal alignment of a railway line, the geometric arrangement of reverse curves consists of three successive main directions (referred to as *Line 1*, *Line 2*, and *Line 3*), with the last direction (i.e., *Line 3*) deviating from the preceding direction (*Line 2*) in the opposite direction from before. These intersect at points W_1 (*Line 1* and *Line 2*) and W_2 (*Line 2* and *Line 3*). The layout of the main directions is transformed from the PL-2000 coordinate system into a local coordinate system using the transformation equations presented in [16]. This may result in the layouts shown in Figure 1. It presents the considered layouts of the main routes, which differ in the position of *Line 3*; this is due to the varying distances between the main points W_1 and W_2 . The purpose is to verify the suggestion made in [16] that this distance must not be too short. In Layout I, the distance $W_1W_{2(I)} = 1000$ m; in Layout II, $W_1W_{2(II)} = 400$ m; and in Layout III, $W_1W_{2(III)} = 200$ m.

The inclinations of the individual main directions are as follows: for *Line 1* $\varphi_1 = 0.523599$ rad; for

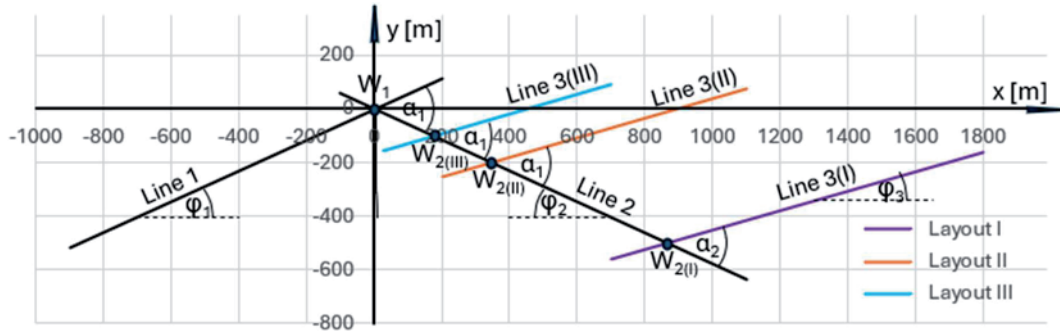


Fig. 1. Layouts of the main directions of the route under consideration, with varying positions of the third main direction [Author's own work]

Line 2 $\varphi_2 = -0.523599$ rad; and for Line 3 (in each of the three cases) $\varphi_3 = 0.349066$ rad. This gives the curve deflection angles: $\alpha_1 = |\varphi_2 - \varphi_1| = 1.047198$ rad, $\alpha_2 = |\varphi_3 - \varphi_2| = 0.872664626$ rad. The coordinates of point W_1 are $x_{W_1} = 0$, $y_{W_1} = 0$, while the coordinates of point W_2 vary:

- for Layout I: $x_{W_2(I)} = 866.025$ m, $y_{W_2(I)} = -500$ m,
- for Layout II: $x_{W_2(II)} = 346.410$ m, $y_{W_2(II)} = -200$ m,
- for Layout III: $x_{W_2(III)} = 173.205$ m, $y_{W_2(III)} = -100$ m.

The presented main direction layouts are the basis for further detailed analysis.

3. Significant Distance between the Main Points of the Reverse Curve (Layout I)

3.1. Application of Circular Curves (Variant IA)

As shown in [16], in the geometric layout of reverse curves, there is a relationship between the radii of the two curves. These values must correspond to the existing layout of the main directions. The key issue here is selecting the correct connection point for the circular curves. This point (labeled B in Figure 2)

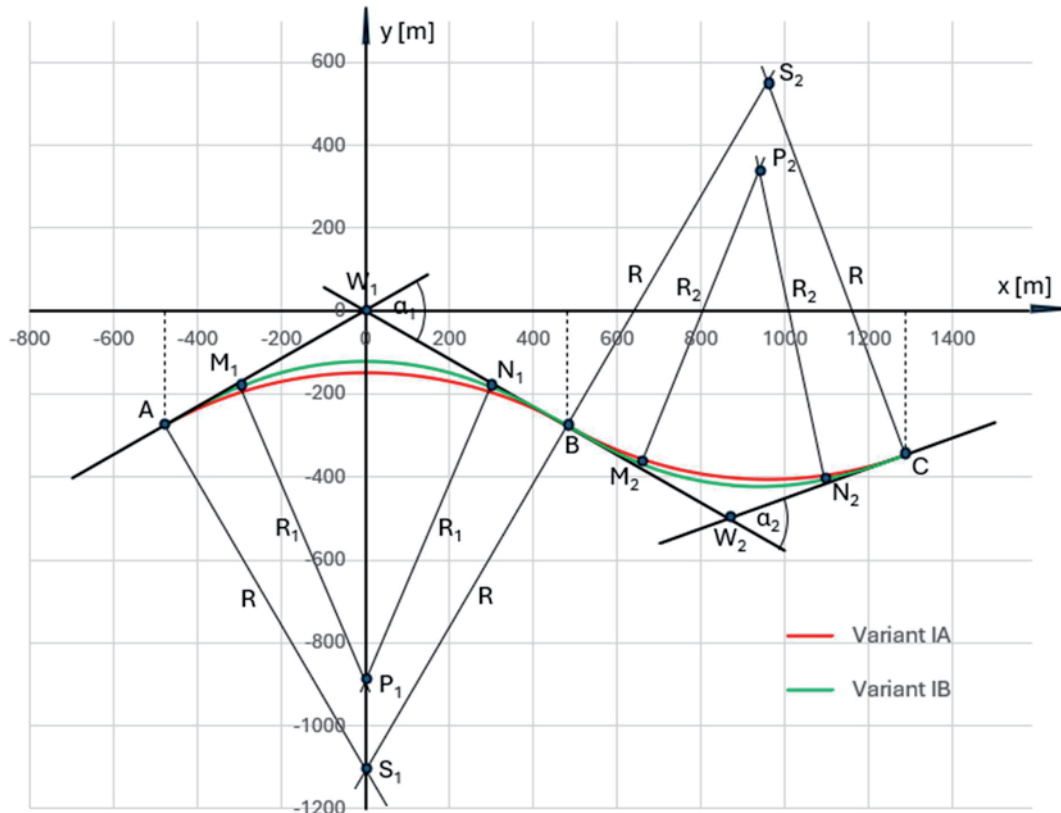


Fig. 2. Determined standard geometric layouts of reverse curves for Layout I ($\overline{W_1W_2} = 1000$ m) [Author's own work]

Table 1

Final part of the calculations for determining the original radius of the reverse curves

$\overline{W_1B}$ [m]	α_1 [rad]	R_1 [m]	$\overline{BW_2}$ [m]	α_2 [rad]	R_2 [m]	ΔR [m]
553.1984	1.047198	958.1677	446.8016	0.872664626	958.1691	0.001388
553.1985	1.047198	958.1679	446.8015	0.872664626	958.1689	0.001000
553.1986	1.047198	958.1681	446.8014	0.872664626	958.1687	0.000613
553.1987	1.047198	958.1683	446.8013	0.872664626	958.1685	0.000225
553.1988	1.047198	958.1684	446.8012	0.872664626	958.1683	-0.000163
553.1989	1.047198	958.1686	446.8011	0.872664626	958.1681	-0.000550
553.1990	1.047198	958.1688	446.8010	0.872664626	958.1678	-0.000938

[Author's own work].

lies on *Line 2*, and the conditions $x_B \in (x_{W_1}, x_{W_2})$ and apply $\overline{W_1B} + \overline{BW_2} = \overline{W_1W_2}$. For a given layout, the position of point *B* determines the values of both radii, since the values of $\overline{W_1B}$ and $\overline{BW_2}$ are the tangents of the respective curves. Generally speaking, increasing the radius of one curve makes it necessary to reduce the radius of the reverse curve.

It was determined that, from the perspective of achieving the highest possible train speed, the most favorable configuration would be one in which, in the original layout (i.e., without the use of transition curves), both circular curves have the same radius. This value can be determined iteratively by assuming different positions of point *B* on *Line 2* (more precisely, different distances of this point from the point W_1). The equations for the radii given in [16] were used.

$$R_1 = \frac{\sqrt{(x_B - x_{W_1})^2 + (y_B - y_{W_1})^2}}{\tan \frac{|\varphi_2 - \varphi_1|}{2}} = \frac{\overline{W_1B}}{\tan \frac{\alpha_1}{2}} \quad (1)$$

$$R_2 = \frac{\sqrt{(x_{W_2} - x_B)^2 + (y_{W_2} - y_B)^2}}{\tan \frac{|\varphi_3 - \varphi_2|}{2}} = \frac{\overline{BW_2}}{\tan \frac{\alpha_2}{2}} \quad (2)$$

Table 1 presents the final part of the calculations.

As shown, in the case under consideration, the smallest difference $\Delta R = R_2 - R_1$ occurs when $\overline{W_1B} = 553.1988$ m; therefore, the radii of the reverse curves determined iteratively are $R = R_1 = R_2 = 958.168$ m. Since $s_2 = \tan \varphi_2 = -0.57735$, the coordinates of point *B* are $x_B = 479.084$ m, $y_B = -276.599$ m. The coordinates of the starting point *A* follow directly from this:

$x_A = -479.084$ m, $y_A = -276.599$ m. Meanwhile, the distance $\overline{BW_2} = \overline{W_1W_2} - \overline{W_1B}$ is 446.8012 m, which gives the coordinates of point *C*: $x_C = 1285.881$ m, $y_C = -347.185$ m.

Inserting reverse curves of radius *R* into the geometric layout yields the result shown in Figure 2 (marked in red). The coordinates of the centers of these curves are as follows: $x_{S1} = 0$, $y_{S1} = -1106.398$ m and $x_{S2} = 958.168$ m, $y_{S2} = 553.199$ m.

The discussed reverse curve layout (referred to as the original layout) has the curvature graph shown in Figure 3 (also marked in red).

As shown, the original configuration features abrupt changes in curvature. In actual operation, of course, they do not exist, as they are absorbed along the length of the rail vehicle's rigid base. In the case of cant, which in the adopted model is directly related to the track curvature, structural constraints preclude abrupt changes in its value. Consequently, in this case, cant cannot be provided on the circular curves. As a result, train operation must be expected at relatively low speeds.

The maximum speed of trains on a given geometric layout is determined using the following general equations:

$$V_{\max} = \min(V^{(a)}, V^{(f)}, V^{(\psi)}) \quad (3)$$

$$V^{(a)} = 3,6 \sqrt{\frac{1}{\Delta \kappa} \cdot \left(a_{\text{per}} + g \frac{\Delta h}{s} \right)} \quad (4)$$

$$V^{(f)} = \frac{3,6 \cdot \Delta l \cdot f_{\text{per}}}{\Delta h} \quad (5)$$

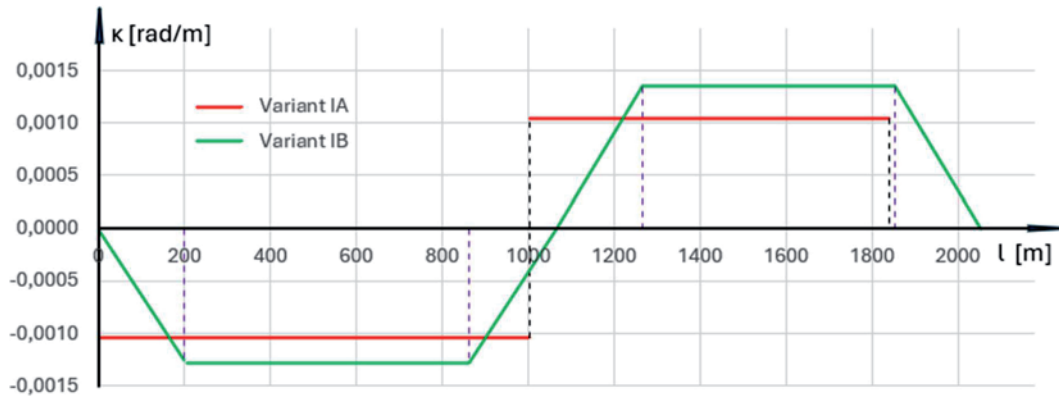


Fig. 3. Curvature graphs along the length of the geometric layout shown in Figure 2 [Author's own work]

$$V^{(\psi)} = \frac{3,6 \cdot \Delta l \cdot \psi_{per}}{\Delta a} \quad (6)$$

where:

$\Delta \kappa$ – is the value of the horizontal curvature difference in rad/m,

Δh – is the value of the cant difference in millimeters,

Δa – is the value of the unbalanced acceleration difference in m/s^2 ,

$$\Delta a = \left[\frac{\min(V^{(a)}, V^{(f)})}{3,6} \right]^2 \cdot \Delta \kappa - g \frac{\Delta h}{s} \quad (7)$$

Δl – is the length of the track section where $\Delta \kappa / \Delta l$, $\Delta h / \Delta l$, and $\Delta a / \Delta l$ have the maximum value (in meters),

a_{per} – is the permissible value of unbalanced acceleration on a circular curve; $a_{per} = 0.85 \text{ m/s}^2$ was adopted [27],

f_{per} – is the permissible rate of wheel lift on the gradient due to cant; $f_{per} = 28 \text{ mm/s}$ was adopted [24],

ψ_{per} – is the permissible rate of change of acceleration on the transition curve; $\psi_{per} = 0.5 \text{ m/s}^3$ was adopted [24],

g – is the acceleration due to gravity ($g = 9.81 \text{ m/s}^2$),

s – is the center distance of rails ($s = 1500 \text{ mm}$).

For circular curves, $\Delta \kappa = 1/R$ and $\Delta h = h_0$, where h_0 denotes the cant on the circular curve (in millimeters). In the present analysis, it is assumed that $\Delta l = l_b = 20 \text{ m}$ (corresponding to the rigid base of the rail vehicle). As follows from the above equations, for the adopted permissible kinematic parameters, the speed $V^{(a)}$ depends on the curve radius R and the cant h_0 , the speed $V^{(f)}$ depends on h_0 and Δl , while the

speed $V^{(\psi)}$ depends on Δa and Δl . It should also be noted that increasing the cant h_0 increases the permissible speed $V^{(a)}$; however, it simultaneously reduces the permissible speed $V^{(f)}$. This trade-off is therefore an important consideration in determining $V^{(\psi)}$.

In this case, there is no cant on the circular curves (that is, $\Delta h = h_0 = 0$). Using Equation (4), $V^{(a)}$ was calculated to be 102.738 km/h, and using Equation (6), $V^{(\psi)}$ was calculated to be 42.353 km/h (the speed $V^{(f)}$ is not relevant here). The maximum achievable speed is therefore $V_{\max} = 42 \text{ km/h}$ and is relatively low; it thus becomes necessary to introduce transition curves.

3.2. Application of Circular Curves with Transition Curves (Variant IB)

The relevant procedure concerning the application of circular curves with transition curves was described in detail in [16]. The designed geometric layout for Variant IB is shown in Figure 2 (marked in green). The purpose of introducing transition curves was to achieve the highest possible train speeds. Accordingly, a cant of $h_{\max} = 150 \text{ mm}$ was adopted for both curves. Clothoid transition curves were employed, with their lengths adjusted to the radii of the reverse curves, R_1 and R_2 , determined iteratively. These radii are smaller than the original radius R and differ from one another because of the different curve deflection angles. The characteristic points shown in Figure 2 are defined as follows:

- A – is the starting point of the layout: the beginning of the first original curve of radius R and of the first transition curve (with a length of l_{11}) of the circular curve of radius R_1 ,
- M_1 – is the end of the first transition curve of the circular curve of radius R_1 ,
- N_1 – is the end of the second transition curve (with a length of l_{12}) of the circular curve of radius R_1 ,

- B – is the point where the two reverse curves meet: the end of the first original curve and the beginning of the second transition curve of the circular curve with radius R_1 as well as the beginning of the second (reverse) original curve and the beginning of the first transition curve (with a length of l_{21}) of the curve with radius R_2 ,
- M_2 – is the end of the first transition curve of the circular curve of radius R_2 ,
- N_2 – is the end of the second transition curve (with a length of l_{22}) of the circular curve of radius R_2 ,
- C – is the endpoint of the layout: the end of the second original curve and the beginning of the second transition curve of the circular curve of radius R_2 .

The radii R_1 and R_2 of the two reverse curves are determined iteratively, taking into account the curve deflection angle for different lengths of transition curves. It was assumed that symmetrical layouts would be formed, in which $l_{11} = l_{12} = l_1$ and $l_{21} = l_{22} = l_2$. Table 2 presents the final part of the calculations for the value of the radius R_1 , assuming $l_1 = 205$ m. The values Δx_{TC} and Δx_{CA} represent the lengths of the projections onto the x -axis of the transition curve and half of the circular curve, respectively; they are determined using the following equations:

$$\Delta x_{TC} = |x_{M1} - x_A|, \quad \Delta x_{CA} = \frac{|s_{M1}|}{\sqrt{1 + s_{M1}^2}} R_1.$$

In this case, the criterion for selecting the appropriate radius is the condition:

$$\Delta x = \Delta x_{TC} + \Delta x_{CA} + x_A \approx 0.$$

Table 2 shows that the smallest difference Δx occurs for the radius $R_1 = 778.488$ m. Determining the final values of l_1 and R_1 requires a speed analysis, as shown in Table 3.

Based on the data in Table 3, it can be concluded that the speed criterion $V^{(\psi)}$ is of no use in this case; therefore, only the criteria $V^{(f)}$ and $V^{(a)}$ apply. It follows from these that for the length $l_1 = 200$ m, the speed $V_{\max} = \min(V^{(f)}, V^{(a)}) = 134$ km/h, while for $l_1 = 205$ m and $l_1 = 210$ m, it is 135 km/h. To avoid unnecessarily extending the entire geometric layout, the length of the transition curve should be set to $l_1 = 205$ m and the radius of the circular curve to $R_1 = 778.488$ m.

By following the procedure outlined in the scheme, a set of numerical data was obtained, which made it possible to create the configuration shown in green in Figure 2. The radii of the circular curves are $R_1 = 778.488$ m, $R_2 = 741.578$ m, and the coordinates of

Table 2

Final part of the calculations in determining the curve radius with transition curves

R_1 [m]	α_1 [rad]	x_A [m]	x_{M1} [m]	s_1 [rad]	Δx_{TC} [m]	Δx_{CA} [m]	Δx [m]
778.492	1.047198	-479.084	-297.364	0.41331746	181.7207	297.3656	0.0020607
778.491	1.047198	-479.084	-297.364	0.41331726	181.7207	297.3651	0.0015620
778.490	1.047198	-479.084	-297.364	0.41331706	181.7207	297.3646	0.0010633
778.489	1.047198	-479.084	-297.364	0.41331686	181.7207	297.3641	0.0005646
778.488	1.047198	-479.084	-297.364	0.41331667	181.7207	297.3636	0.0000659
778.487	1.047198	-479.084	-297.364	0.41331647	181.7207	297.3631	-0.000433
778.486	1.047198	-479.084	-297.364	0.41331627	181.7207	297.3626	-0.000931

[Author's own work].

Table 3

Determining the values of l_1 and R_1 based on speed criteria

l_1 [m]	$h_{\max}$ [mm]	F_{per} [mm/s]	$V^{(f)}$ [km/h]	R_1 [m]	a_{per} [m/s ²]	$V^{(a)}$ [km/h]	Ψ_{per} [m/s ³]	$V^{(\psi)}$ [km/h]
200	150	28	134.40	782.930	0.85	136.304	0.5	423.529
205	150	28	137.76	778.488	0.85	135.917	0.5	444.706
210	150	28	141.12	774.042	0.85	135.528	0.5	465.882

[Author's own work].

their centers are as follows: $x_{P1} = 0$, $y_{P1} = -893.564$ m, and $x_{P2} = 939.500$ m, $y_{P2} = 339.814$ m. Symmetrical transition curves of lengths $l_1 = 205$ m and $l_2 = 200$ m were used. The coordinates of the characteristic points A, B, and C are the same as before, while at the ends of the transition curves, they are $x_{M1} = -297.364$ m, $y_{M1} = -182.059$ m, $x_{N1} = 297.364$ m, $y_{N1} = -182.059$ m, $x_{M2} = 656.464$ m, $y_{M2} = -368.642$ m and $x_{N2} = 1095.213$ m, $y_{N2} = -407.028$ m.

The curvature graph along the length of the geometric layout with transition curves (Figure 2) is shown in Figure 3 (marked in green).

In the case under consideration, the maximum speed of trains on a given layout of reverse curves is determined by the smaller curve radius (i.e., radius R_2). The value of $\Delta\kappa$ in Equation (4) is $\Delta\kappa = 1/R_2$, Δh in Equations (4) and (5) is $\Delta h = h_2$ and Δl in Equations (5) and (6) is $\Delta l = l_2$. For the radius $R_2 = 741.578$ m, the cant applied on this curve $h_2 = 150$ mm, and the transition-curve length $l_2 = 200$ m, the speed $V^{(a)} = 132.655$ km/h, $V^{(f)} = 134.400$ km/h, and $V^{(\psi)} = 423.529$ km/h. The maximum speed achievable is therefore $V_{\max} = 132$ km/h. As can be seen, when

there is a significant distance between the main points along the route, reverse curves do not significantly limit train speeds.

4. Limited Distance between the Main Points of the Reverse Curve (Layout II)

4.1. Application of Circular Curves (Variant IIA)

Using the same procedure as before, the radii of the original reverse curves for Layout II, where $W_1W_2 = 400$ m, were determined iteratively, yielding $R = 383.267$ m. The coordinates of the characteristic points are as follows: $x_A = -191.634$ m, $y_A = -110.640$ m, $x_B = 191.634$ m, $y_B = -110.640$ m, $x_C = 514.353$ m, $y_C = -138.874$ m. The reverse curves of radius R inserted into the geometric layout are shown in Figure 4 (marked in red). The coordinates of the centers of these curves are as follows: $x_{S1} = 0$, $y_{S1} = -442.559$ m and $x_{S2} = 383.267$ m, $y_{S2} = 221.280$ m.

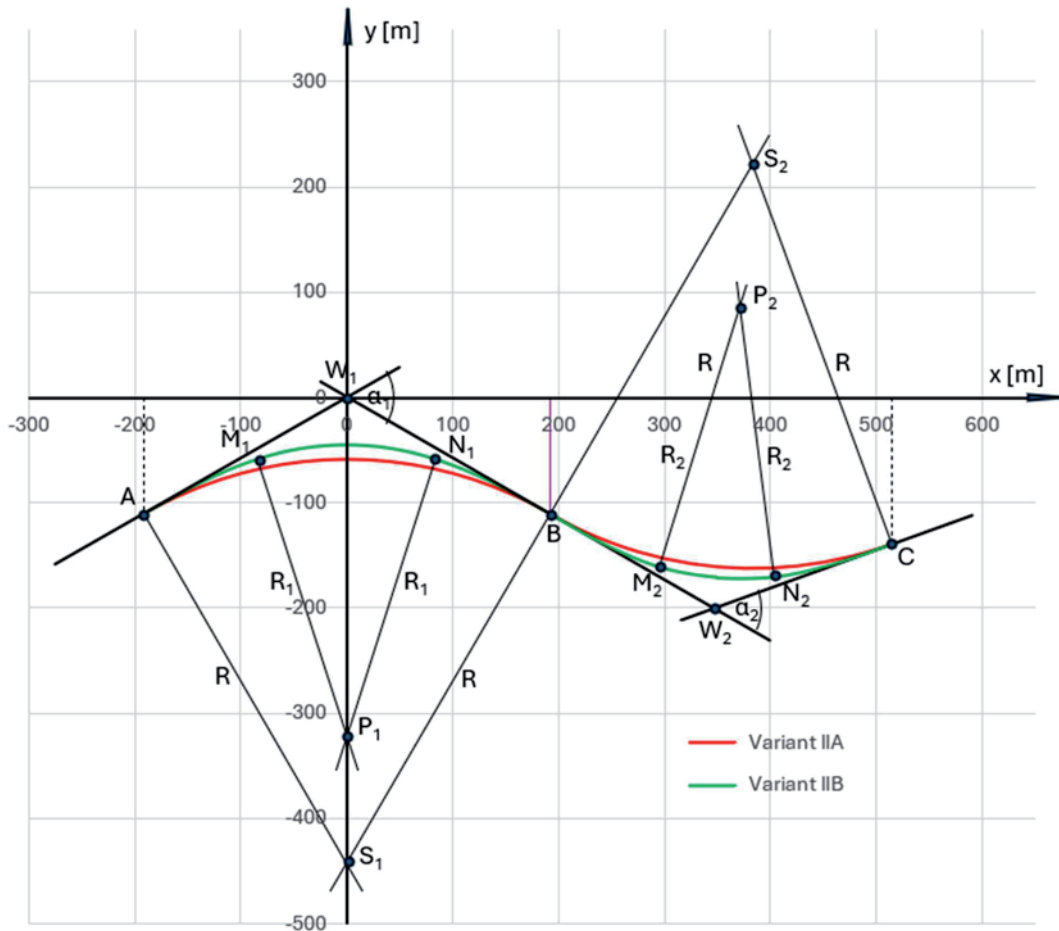


Fig. 4. Determined standard geometric layouts of reverse curves for Layout II ($W_1W_2 = 400$ m) [Author's own work]

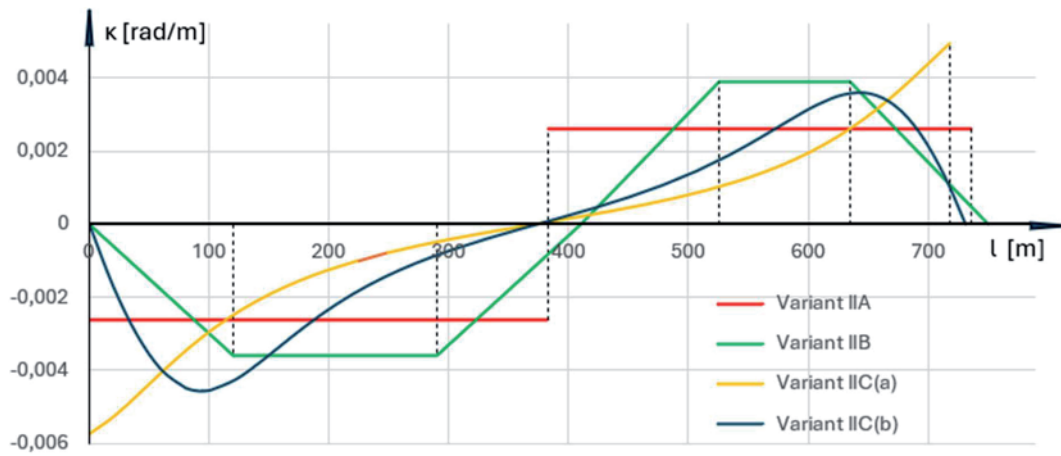


Fig. 5. Curvature graphs along the length of standard geometric layouts and the curves connecting points A and C in Layout II [Author's own work]

The curvature graph along the length of this geometric layout is shown in Figure 5 (also marked in red). As in the corresponding variant of Layout I, it features abrupt curvature changes, which are crucial for determining the maximum speed that trains can achieve.

As before, in this case, there is no cant on the circular curves (that is, $\Delta h = h_0 = 0$); when determining the speed, $\Delta l = l_b = 20$ m was assumed. Using Equation (4), $V^{(a)}$ was calculated to be 64.977 m, and using Equation (6), $V^{(\psi)}$ was calculated to be 42.353 km/h. The maximum achievable speed $V_{\max} = 42$ km/h should therefore be increased by introducing transition curves.

4.2. Application of Circular Curves with Transition Curves (Variant IIB)

The designed layout with transition curves is also shown in Figure 4 (marked in green). For the adopted cant on both curves, $h = 150$ mm, symmetrical clothoid transition curves with lengths $l_1 = 120$ m and $l_2 = 115$ m were used. The radii of the reverse curves, determined in the same way as before (i.e., iteratively), have decreased compared to the radius R and are as follows: $R_1 = 277.346$ m, $R_2 = 257.969$ m, and the coordinates of their centers are as follows: $x_{P1} = 0$, $y_{P1} = -442.559$ m and $x_{P2} = 371.434$ m, $y_{P2} = 86.029$ m. The coordinates of the characteristic points A, B, and C are the same as before, while at the ends of the transition curves they are $x_{M1} = -191.634$ m, $y_{M1} = -110.640$ m, $x_{N1} = 191.634$ m, $y_{N1} = -110.640$ m, $x_{M2} = 294.990$ m, $y_{M2} = -160.481$ m, and $x_{N2} = 403.912$ m, $y_{N2} = -170.010$ m.

The curvature graph along the length of the geometric layout with transition curves (Figure 4) is shown in Figure 5 (marked in green).

The maximum train speed on a given layout of reverse curves is determined by the radius of the smaller curve

(i.e., radius R_2). For the radius $R_2 = 257.969$ m, the cant applied on this curve $h_2 = 150$ mm, and the transition-curve length $l_2 = 115$ m, the speed $V^{(a)} = 78.240$ km/h, $V^{(f)} = 77.280$ km/h, and $V^{(\psi)} = 243.529$ km/h. The maximum speed achievable is therefore $V_{\max} = 77$ km/h. This represents a significant limitation compared to the corresponding situation in Layout I. An attempt was therefore made to deviate from the standard solution to the problem and to determine a curve that directly connects points A and C.

4.3. Application of Direct Connection of Endpoints (Variant IIC)

The problem under consideration can be solved in a completely different way. The desired connection between endpoints A and C does not require point B to be included; it may be omitted. The coordinates of points A and C for the case under consideration were determined in Subsection 4.1. The values of the inclination of the tangent at these points are also known: $s_A = s_1 = 0.57735$, $s_C = s_3 = 0.36397$. In order to simplify the procedure for determining the connecting curve, an auxiliary coordinate system $\bar{x}$, $\bar{y}$, associated with this curve, was introduced. This system is created after rotating the x , y system clockwise by an angle of $\varphi_1 = 0.523599$ rad and shifting the origin to the left by AW_1 . The following transformation equations were used:

$$\bar{x} = x \cdot \cos \varphi_1 + y \cdot \sin \varphi_1 + \overline{AW_1}, \quad (8)$$

$$\bar{y} = -x \cdot \sin \varphi_1 + y \cdot \cos \varphi_1, \quad (9)$$

where $\overline{AW_1} = \sqrt{(x_{W1} - x_A)^2 + (y_{W1} - y_A)^2} = \bar{x}_{W1}$.

In the $\bar{x}, \bar{y}$ system, the conditions relating to point A (i.e., for $\bar{x} = 0$) are as follows:

$$\bar{x}_A = 0, \bar{y}_A = 0, \bar{s}_A = 0,$$

for point C:

$$\bar{x}_C = x_C \cdot \cos \varphi_1 + y_C \cdot \sin \varphi_1 + \overline{AW_1},$$

$$\bar{y}_C = -x_C \cdot \sin \varphi_1 + y_C \cdot \cos \varphi_1, \bar{s}_C = \tan(\varphi_3 - \varphi_1).$$

Additional notation was also introduced:

$$\Delta \bar{x} = \bar{x}_C - \bar{x}_A = \bar{x}_C, \Delta \bar{y} = \bar{y}_C - \bar{y}_A = \bar{y}_C.$$

The elementary curve connecting the points should satisfy the following boundary conditions:

$$\begin{cases} \bar{y}(0) = 0 & \bar{y}(\Delta \bar{x}) = \Delta \bar{y} \\ \bar{y}'(0) = 0 & \bar{y}'(\Delta \bar{x}) = \bar{s}_C \end{cases} \quad (10)$$

and the differential equation (for the polynomial solution):

$$\bar{y}^{(4)}(\bar{x}) = 0. \quad (11)$$

The general solution to the differential problem (10), (11) is a third-degree polynomial:

$$\bar{y}(\bar{x}) = a_1 \cdot \bar{x}^2 + a_2 \cdot \bar{x}^3 \quad (12)$$

$$\text{Where: } a_1 = -\frac{\bar{s}_C \cdot \Delta \bar{x} - 3 \cdot \Delta \bar{y}}{\Delta \bar{x}^2}, a_2 = \frac{\bar{s}_C \cdot \Delta \bar{x} - 2 \cdot \Delta \bar{y}}{\Delta \bar{x}^3}.$$

In the case under consideration, $\Delta \bar{x} = 597.285$ m, $\Delta \bar{y} = -377.444$ m, $\bar{s}_C = -0.176327$; the obtained coefficients of Equation (12) are $a_1 = -0.0028788$ 1/m, $a_2 = 3.04847\text{E-}06$ 1/m². The connecting curve (designated as Variant IIC(a)) – compared to the standard solutions from Variants IIA and IIB – is shown in Figure 6 (marked in orange).

Based on Figure 6, it would be difficult to evaluate the connecting curve in Variant IIC(a) and compare it with the solutions in Variant IIA and Variant IIB. A suitable criterion here may be the curvature graph along the length. The general equation for the curvature $\kappa(\bar{x})$ of a given curve is as follows:

$$\kappa(\bar{x}) = \frac{|\bar{y}''(\bar{x})|}{\left\{ \sqrt{1 + [\bar{y}'(\bar{x})]^2} \right\}^3} \quad (13)$$

$$\text{where: } \bar{y}'(\bar{x}) = 2 \cdot a_1 \cdot \bar{x} + 3 \cdot a_2 \cdot \bar{x}^2,$$

$$\bar{y}''(\bar{x}) = 2 \cdot a_1 + 6 \cdot a_2 \cdot \bar{x}.$$

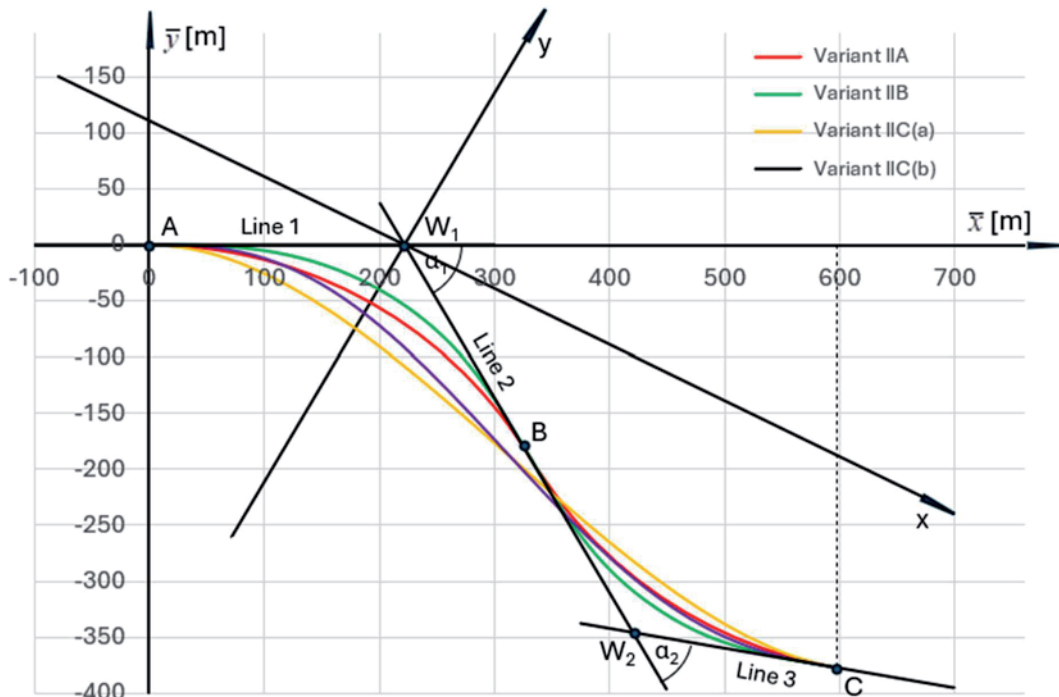


Fig. 6. The connecting curves in Variants IIC(a) and IIC(b) compared to standard geometric layouts [Author's own work]

To obtain the curvature values along the length, the Curve Arc Length Calculator [10] was used, which employed Simpson's rule for integration. The $\kappa(l)$ graph for Variant IIC(a) is shown in Figure 5 (marked in orange). As shown, there are abrupt curvature changes at the starting point and the endpoint; therefore, this solution is not recommended. The maximum absolute value of the curvature occurs at $l = 0$ and is $\kappa = 0.0057576$ rad/m. This is where the maximum cant should occur, which is technically impossible (there cannot be a sudden change in cant on the actual track). Therefore, in the geometric layout discussed, the situation is the same as in Variant IIA.

Based on the curvature value at the starting point, using Equation (4), it follows that $V^{(a)} = 43.741$ km/h. In the context of speed $V^{(\psi)}$, the ratio $\Delta\kappa/\Delta l$ is significant. Its maximum value occurs at $l \in \langle 40.319; 61.015 \rangle$ m. This means that $\Delta\kappa = 0.0006425$ rad/m, so in Equation (6) one should substitute $\Delta a = 0.0948$ m/s² and $\Delta l = 20.696$ m; this gives $V^{(\psi)} = 392.776$ km/h. The maximum speed achievable in Variant IIC(a) is $V_{\max} = 43$ km/h. Under these circumstances, this variant does not offer a favorable solution to the problem.

Given that the boundary conditions (10) do not include the curvature values at points A and C, the decision was made to modify these conditions accordingly. To account for the curvature of the track axis, Equation (13) must be used in the following form:

$$\bar{y}''(\bar{x}) = \left\{ \sqrt{1 + [\bar{y}'(\bar{x})]^2} \right\}^3 \kappa(\bar{x}) \quad (14)$$

At points A and C the values $\kappa(\bar{x})$ are zero; therefore – based on Equation (14) – the second derivatives of the function $\bar{y}(\bar{x})$ are zero at those points. This leads to a correction of the conditions (10); for the connecting curve defined as Variant IIC(b), these take the form:

$$\begin{cases} \bar{y}(0) = 0 & \bar{y}(\Delta\bar{x}) = \Delta\bar{y} \\ \bar{y}'(0) = 0 & \bar{y}'(\Delta\bar{x}) = \bar{s}_c \\ \bar{y}''(0) = 0 & \bar{y}''(\Delta\bar{x}) = 0 \end{cases} \quad (15)$$

In this situation, the following differential equation should be used:

$$\bar{y}^{(6)}(\bar{x}) = 0. \quad (16)$$

The solution to the differential problem (15), (16) is a fifth-degree polynomial:

$$\bar{y}(\bar{x}) = a_1 \cdot \bar{x}^3 + a_2 \cdot \bar{x}^4 + a_3 \cdot \bar{x}^5 \quad (17)$$

$$\text{where: } a_1 = \frac{10 \cdot \Delta\bar{y} - 4 \cdot \bar{s}_c \cdot \Delta\bar{x}}{\Delta\bar{x}^3},$$

$$a_2 = -\frac{15 \cdot \Delta\bar{y} - 7 \cdot \bar{s}_c \cdot \Delta\bar{x}}{\Delta\bar{x}^4}, \quad a_3 = \frac{6 \cdot \Delta\bar{y} - 3 \cdot \bar{s}_c \cdot \Delta\bar{x}}{\Delta\bar{x}^5}.$$

In the case under consideration, the coefficients of polynomial (17) are as follows: $a_1 = -1.57366\text{E-}05$ 1/m², $a_2 = 3.86929\text{E-}08$ 1/m³, $a_3 = -2.56355\text{E-}11$ 1/m⁴. The determined connecting curve in Variant IIC(b) – compared to the standard solutions and the curve from Variant IIC(a) – is shown in Figure 6 (marked in purple).

To determine the curvature, Equation (13) should be used, substituting the derivative values

$$\bar{y}'(\bar{x}) = 3 \cdot a_1 \cdot \bar{x}^2 + 4 \cdot a_2 \cdot \bar{x}^3 + 5 \cdot a_3 \cdot \bar{x}^4,$$

$$\bar{y}''(\bar{x}) = 6 \cdot a_1 \cdot \bar{x} + 12 \cdot a_2 \cdot \bar{x}^2 + 20 \cdot a_3 \cdot \bar{x}^3.$$

The $\kappa(l)$ graph for Variant IIC(b) is shown in Figure 5 (marked in purple). The values of $\kappa(l)$ directly determine the values of the cant $h(l)$ based on the relationship

$$h(l) = \frac{h_0}{\kappa_{\max}} \kappa(l) \quad (18)$$

The maximum absolute value of curvature occurs at $l_0 = 93.893$ m and is $\kappa_{\max} = 0.0045743$ rad/m. When determining the speed $V^{(a)}$ using Equation (4), one should assume $\Delta\kappa = \kappa_{\max}$ and $\Delta h = h_0$ (where h_0 denotes the value of the cant at the point where the maximum curvature occurs). The speed $V^{(a)}$ increases as h_0 increases.

By contrast, the maximum value of $\Delta\kappa/\Delta l$, relevant from the point of view of speeds $V^{(f)}$ and $V^{(\psi)}$, occurs at the beginning of the connecting curve for $l \in \langle 0; 20.0006 \rangle$ m. When determining the speed $V^{(f)}$ using Equation (5), one should assume $\Delta\kappa = 0.001706$ rad/m and Δh equal to the cant determined using Equation (18) for $l = \Delta l = 20.0006$ m. The speed $V^{(f)}$ decreases as the adopted h_0 increases.

In the case under consideration, similar $V^{(a)}$ and $V^{(f)}$ were obtained for $h_0 = 85$ mm, i.e., $V^{(a)} = 63.113$ km/h and $V^{(f)} = 63.597$ km/h. Since Equation (7) gives $\Delta a = 0.317$ m/s², the speed $V^{(\psi)} = 113.567$ km/h. In Variant IIC(b), the maximum speed achievable $V_{\max} = 63$ km/h. It is significantly lower compared to circular curves with transition curves (i.e., in Variant IIB). Therefore, the presented solution cannot be recommended for practical application.

4.4. Application of an Extended Connecting Curve (Variant IID)

However, the concept of determining a connecting curve, as presented in Subsection 4.3 and relating to the solution using Equation (17), can be extended. After all, this curve does not have to connect points A and C, which result from the positions of the inserted circular curves. These may be other selected points located on *Line 1* and *Line 3*. There is also no doubt that the new curve should be extended. This extension may result from shifting point A to the left by a distance ΔL_A , shifting point C to the right by a distance ΔL_C , or shifting both points simultaneously (which should be the most advantageous). Since the values of ΔL_A and ΔL_C are in principle arbitrary (or, more precisely, limited by the given geometric configuration), the number of solutions may be enormous. The determined connecting curves will be denoted as $CC(i)$, $i = 0, 1, 2, \dots$; Variant IIC(b) is also included as curve CC0.

In the x, y coordinate system, for a given $\Delta L_{A(i)}$, the new starting point $A(i)$ on *Line 1* has the following coordinates:

$$x_{A(i)} = x_A - \Delta L_{A(i)} \cdot \cos \varphi_1 \quad (19)$$

$$y_{A(i)} = y_A + \Delta L_{A(i)} \cdot \sin \varphi_1, \quad (20)$$

whereas for a given $\Delta L_{C(i)}$, the coordinates of the corresponding endpoint $C(i)$ on *Line 3* are as follows:

$$x_{C(i)} = x_C + \Delta L_{C(i)} \cdot \cos \varphi_3 + \Delta L_{A(i)} \cdot \cos \varphi_1 \quad (21)$$

$$y_{C(i)} = y_C + \Delta L_{C(i)} \cdot \sin \varphi_3 \quad (22)$$

In the case under consideration, uniform values of $\Delta L_{A(i)}$ and $\Delta L_{C(i)}$ were assumed, i.e., $\Delta L_{A(i)} = \Delta L_{C(i)} = \Delta L_{(i)}$. The curve connecting CC1 was determined by adopting $\Delta L_1 = 50$ m. The coordinates of the endpoints of this curve in the x, y system are $x_{A1} = -234.935$ m, $y_{A1} = -135.640$ m and $x_{C1} = 514.353$ m, $y_{C1} = -138.874$ m.

For the extended connecting curve, an auxiliary coordinate system $\bar{x}_{(i)}, \bar{y}_{(i)}$ associated with this curve was introduced by rotating the x, y system clockwise by the angle $\varphi_1 = 0.523599$ rad and shifting the starting point to the left by $A_{(i)}W_1$. For the ordinate $\bar{y}_{(i)}$, Equation (9) still applies, while for the abscissa $\bar{x}_{(i)}$, the transformation formula is used:

$$\bar{x}_{(i)} = x \cdot \cos \varphi_1 + y \cdot \sin \varphi_1 + \overline{A_{(i)}W_1}, \quad (23)$$

where

$$\overline{A_{(i)}W_1} = \sqrt{(x_{W1} - x_{A(i)})^2 + (y_{W1} - y_{A(i)})^2} = \bar{x}_{W1}.$$

In the case under consideration, point $A_1(0, 0)$ should be connected with point C_1 with coordinates $\bar{x}_{C1} = \Delta \bar{x} = 696.525$ m, $\bar{y}_{C1} = \Delta \bar{y} = -386.127$ m, while $\bar{s}_{A1} = 0$, $\bar{s}_{C1} = -0.17633$. The coefficients of polynomial (17) are as follows: $a_1 = -9.97287E-06$ 1/m², $a_2 = 2.09552E-08$ 1/m³, $a_3 = -1.188E-11$ 1/m⁴. The determined curve connecting – compared to standard solutions and the connecting curves CC0, CC2, CC3, and CC4 – is shown in Figure 7 (marked in light blue).

The $\kappa(l)$ graph for the connecting curve CC1 is shown in Figure 8 (marked in light blue). The maximum absolute value of curvature occurs at $l_0 = 114.968$ m and is $\kappa_{\max} = 0.0035019$ rad/m. In turn, the maximum value of $\Delta \kappa / \Delta l$ occurs at the beginning of the curve for $l \in \langle 0; 20.0003 \rangle$ m. Similar $V^{(a)}$ and $V^{(f)}$

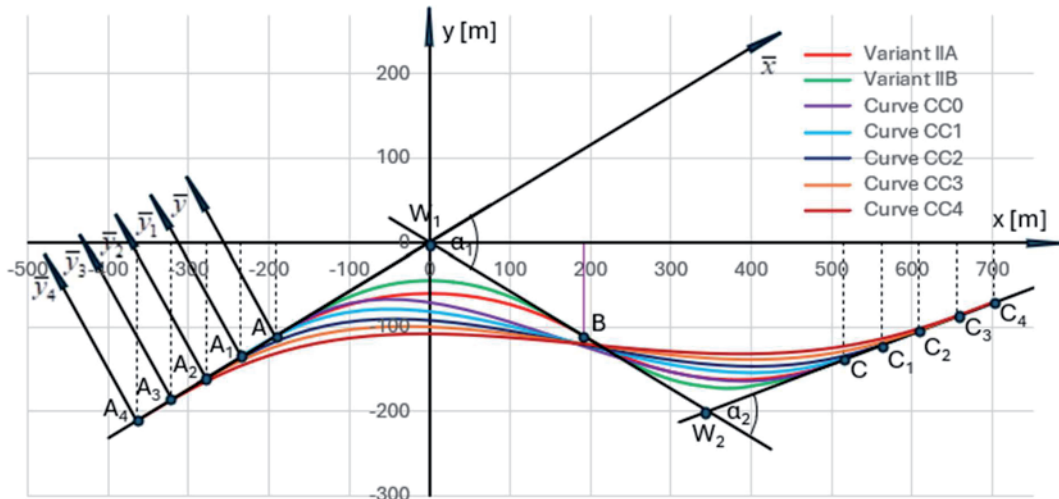


Fig. 7. The determined connecting curves CC0, CC1, CC2, CC3, and CC4 for Layout II, compared to the standard solutions in Variant IIA and Variant IIB [Author's own work]

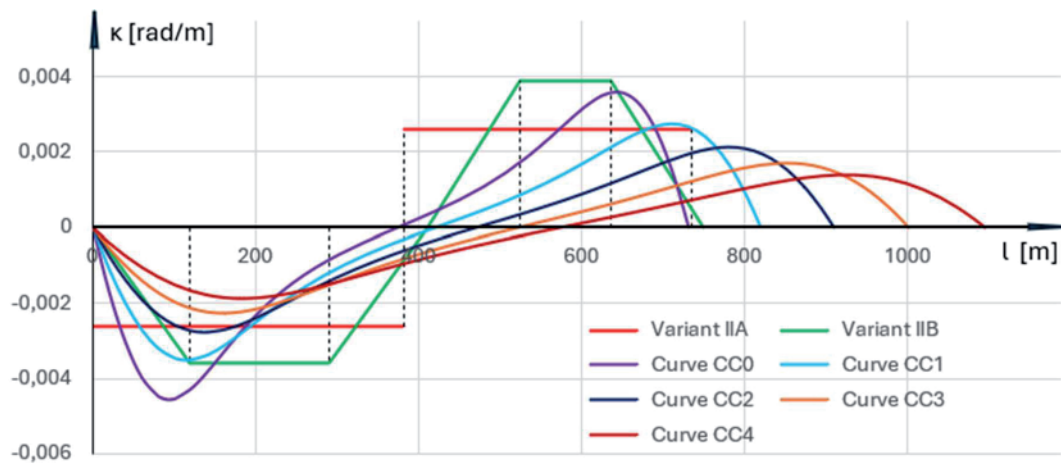


Fig. 8. Curvature graphs along the connecting curves CC0, CC1, CC2, CC3, and CC4 for Layout II, compared to the standard solutions in Variant IIA and Variant IIB [Author's own work]

were obtained for $h_0 = 90$ mm, i.e., $V^{(a)} = 72.966$ km/h and $V^{(j)} = 71.452$ km/h.km/h. Since Equation (7) gives $\Delta a = 0.258$ m/s², the speed $V^{(\psi)} = 139.427$ km/h. On curve CC1, the maximum speed $V_{\max} = 71$ km/h. It is still lower compared to circular curves with transition curves. As shown in Figure 8, the inclination of the initial portion of the curvature graph (and the corresponding gradient due to cant) is more favorable for Variant IIB (i.e., smaller).

The connecting curve was extended further, assuming $\Delta L_2 = 100$ m for curve CC2, $\Delta L_3 = 150$ m for curve CC3, and $\Delta L_4 = 200$ m for curve CC4. Table 4 presents the basic numerical values relating to the determination of the course of the connecting curves (inclinations $\bar{s}_{A(i)} = 0$, $\bar{s}_{C(i)} = -0.17633$).

The determined connecting curves from Table 4 are shown in Figure 7 (alongside the standard solutions). To evaluate these curves, it is necessary to analyze the curvature along their length. The curvature

graphs for the curves from Figure 7 are presented in Figure 8.

Based on the determined curvature (and the corresponding cant), it became possible to determine the speed of trains on a given geometric layout. This is illustrated in Table 5, where $\max(\Delta h/\Delta l)$ denotes the expression in Equation (5). In every case considered, this value occurs in the initial portion of the connecting curve, similarly to $\max \Delta \kappa/\Delta l$.

In the standard solution with transition curves, the speed $V_{\max} = 77$ km/h, and the length of the geometric layout is 750.557 m. The use of a connecting curve with a length of 1,095.950 m permits an increase in speed to $V_{\max} = 100$ km/h. This is undoubtedly a beneficial solution, provided, of course, that no objective obstacles arise (such as those resulting from terrain conditions). There is no doubt, however, that the proposed method for modifying reverse curves offers a good chance of resolving the problems that arise.

Table 4

Basic numerical values for determining connecting curves in Layout II

Connecting curve	$\Delta L_{(i)}$ [m]	$\Delta \bar{x}$ [m]	$\Delta \bar{y}$ [m]	a_1 [1/m ²]	a_2 [1/m ³]	a_3 [1/m ⁴]
CC0	0	597.285	-377.445	-1.57366E-05	3.86929E-08	-2.56355E-11
CC1	50	696.525	-386.127	-9.97287E-06	2.09552E-08	-1.18843E-11
CC2	100	795.766	-394.809	-6.72107E-06	1.23192E-08	-6.10441E-12
CC3	150	895.006	-403.492	-4.74754E-06	7.71077E-09	-3.39117E-12
CC4	200	994.246	-412.174	-3.48022E-06	5.07113E-09	-2.00410E-12

[Author's own work].

Table 5

Calculation of speed for the determined connecting curves in Layout II

Connecting curve	$\Delta L_{(i)}$ [m]	L_{CC} [m]	h_0 [mm]	$\kappa_{\max}$ [rad/m]	$V^{(a)}$ [km/h]	$\max(\Delta h/\Delta l)$ [‰]	$V^{(f)}$ [km/h]	$V^{(\psi)}$ [km/h]	$V_{\max}$ [km/h]
CC0	0	731.037	85	0.0045743	63.113	1.58497	63.598	113.567	63
CC1	50	819.606	90	0.0035019	72.966	1.41074	71.452	139.427	71
CC2	100	910.248	90	0.0027727	82.002	1.21448	82.998	151.116	82
CC3	150	1002.491	95	0.0022534	91.990	1.12399	89.681	180.585	89
CC4	200	1095.950	95	0.0018699	100.981	0.99982	100.818	187.812	100

[Author's own work].

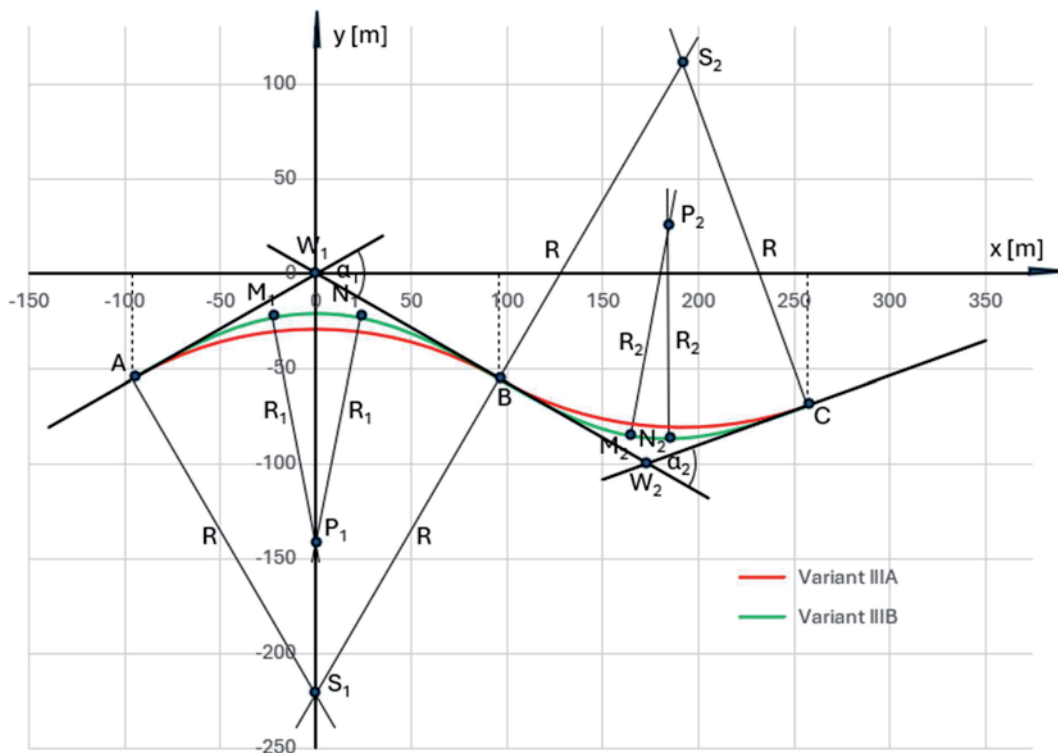
5. Short Distance between the Main Points of the Reverse Curve (Layout III)

5.1. Application of Circular Curves (Variant IIIA)

Using the same procedure, the radii of the original reverse curves for Layout III, where $\overline{W_1W_2} = 200$ m, were determined iteratively, yielding $R = 191.634$ m. The coordinates of the characteristic points are as follows: $x_A = -95.817$ m, $y_A = -55.320$ m, $x_B = 95.817$ m, $y_B = -55.320$ m, $x_C = 257.176$ m, and $y_C = -69.4374$ m.

The reverse curves of radius R inserted into the geometric layout are shown in Figure 9 (marked in red). The coordinates of the centers of these curves are $x_{S_1} = 0$, $y_{S_1} = -221.280$ m and $x_{S_2} = 191.634$ m, $y_{S_2} = 110.639$ m.

The curvature graph along the length of this geometric layout is shown in Figure 10 (also marked in red). Similar to the corresponding variants of Layout I and Layout II, it features abrupt curvature changes. These changes make it impossible to apply cant on the circular curves (that is, $\Delta h = h_0 = 0$). Assuming $\Delta l = l_b = 20$ m, Equation (4) yields $V^{(a)} = 45.946$ m, and

Fig. 9. Determined standard geometric layouts of reverse curves for Layout III ($\overline{W_1W_2} = 200$ m) [Author's own work]

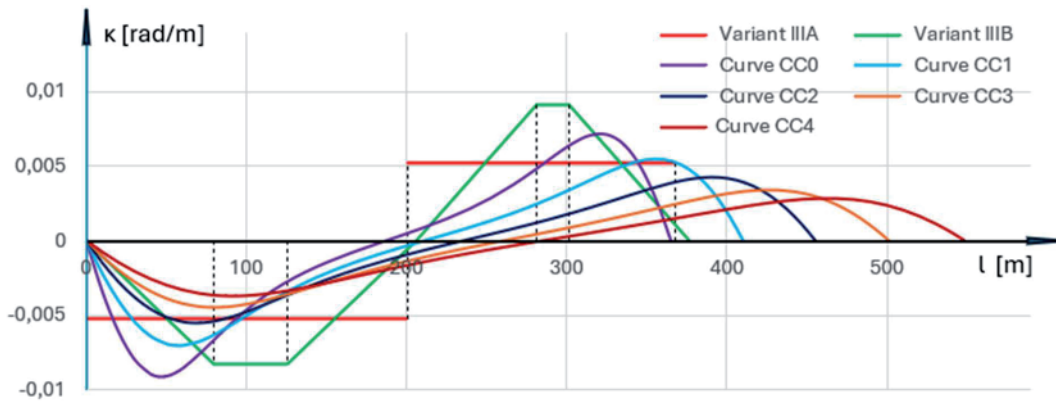


Fig. 10. Curvature graphs along the connecting curves CC0, CC1, CC2, CC3, and CC4 for Layout III, compared to the standard solutions in Variant IIIA and Variant IIIB [Author's own work]

Equation (6) yields $V^{(\psi)} = 42.353$ km/h. It should be determined to what extent the introduction of transition curves would increase the achievable maximum speed $V_{\max} = 42$ km/h.

5.2. Application of Circular Curves with Transition Curves (Variant IIIB)

The designed layout with transition curves is also shown in Figure 9 (marked in green). For the adopted cant on both curves, $h = 150$ mm, symmetrical clothoid transition curves with lengths $l_1 = 80$ m and $l_2 = 75$ m were used. The determined radii of the reverse curves decreased in comparison with radius R and are $R_1 = 120.400$ m, $R_2 = 109.249$ m, and the coordinates of their centers are as follows: $x_{P1} = 0$, $y_{P1} = -141.573$ m and $x_{P2} = 183.976$ m, $y_{P2} = 23.108$ m. Given the standards applicable to railway tracks, these radii are very small and do not meet the design requirements for the wheel-rail system. The coordinates of the characteristic points A, B, and C remain unchanged, while at the ends of the transition curves, they are $x_{M1} = -22.901$ m, $y_{M1} = -23.371$ m, $x_{N1} = 22.901$ m, $y_{N1} = -23.371$ m, $x_{M2} = 164.262$ m, $y_{M2} = -85.012$ m, and $x_{N2} = 184.615$ m, $y_{N2} = -86.793$ m.

The curvature graph along the length of the geometric layout with transition curves (Figure 9) is shown in Figure 10 (marked in green).

The maximum train speed on a given layout is determined by the smaller curve. For the radius $R_2 = 109.249$ m, the cant applied on this curve $h_2 = 150$ mm, and the transition-curve length $l_2 = 75$ m, the speed $V^{(a)} = 58.916$ km/h, $V^{(f)} = 50.400$ km/h, and $V^{(\psi)} = 269.385$ km/h. The maximum speed achievable is therefore $V_{\max} = 50$ km/h. However, the feasibility of operating at such a speed may be called into question for design reasons, i.e., the incorporation of rolling stock wheels into the track with very small radii of the

circular curves. It was therefore decided to connect points A and C directly using a smooth connecting curve (as in Subsection 4.3).

5.3. Application of a Smooth Connecting Curve (Variant IIIC)

The coordinates of points A and C for the case under consideration were determined in Subsection 5.1. The inclinations of the tangent at these points are the same as those used previously. The auxiliary coordinate system $\bar{x}$, $\bar{y}$, is obtained after applying transformation Equations (8) and (9). Boundary conditions (15) and differential Equation (16) apply; the connecting curve is described by Equation (17). In the case under consideration, $\Delta\bar{x} = 298.642$ m, $\Delta\bar{y} = -188.723$ m, $\bar{s}_C = -0.176327$; the obtained coefficients of Equation (17) are $a_1 = -6.29471E-05$ 1/m², $a_2 = 3.09547E-07$ 1/m³, $a_3 = -4.10172E-10$ 1/m⁴. The determined connecting curve CC0 – compared to standard solutions and the extended connecting curves from Subsection 5.4 – is shown in Figure 11 (marked in purple).

The $\kappa(l)$ graph for the CC0 curve is shown in Figure 10 (marked in purple). The maximum absolute value of curvature occurs at $l_0 = 46.426$ m and is $\kappa_{\max} = 0.0091483$ rad/m. In turn, the maximum value of $\Delta\kappa/\Delta l$ occurs at the beginning of the connecting curve for $l \in \langle 0; 10.0003 \rangle$ m. When determining the speed $V^{(f)}$, $\Delta\kappa = 0.003412$ rad/m should be adopted. Similar $V^{(a)}$ and $V^{(f)}$ were obtained for $h_0 = 65$ mm, i.e., $V^{(a)} = 42.501$ km/h and $V^{(f)} = 41.581$ km/h. Since Equation (7) gives $\Delta a = 0.248$ m/s², the speed $V^{(\psi)} = 72.623$ km/h. Thus, by using the curve CC0, the maximum speed $V_{\max} = 41$ km/h can be achieved. It is lower compared to circular curves with transition curves (i.e., in Variant IIIB). Therefore, the presented solution cannot be recommended.

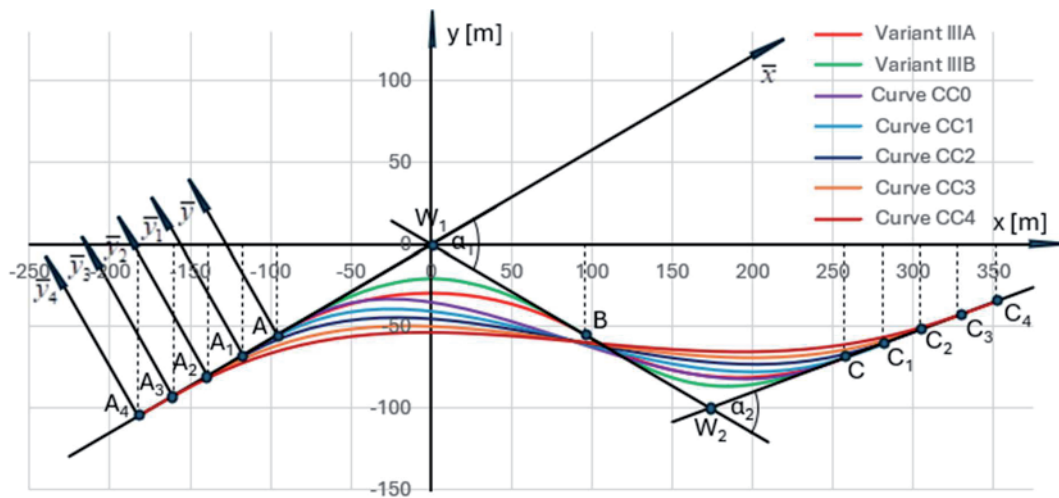


Fig. 11. The determined connecting curves CC0, CC1, CC2, CC3, and CC4 for Layout III, compared to the standard solutions in Variant IIIA and Variant IIIB [Author's own work]

5.4. Application of an Extended Connecting Curve (Variant IIID)

Using the same procedure and assumptions as in Subsection 4.4, the extended connecting curve CC1 for $\Delta L_1 = 25$ m was determined. The coordinates of the endpoints of this curve in the x, y system are $x_{A1} = -117.468$ m, $y_{A1} = -67.820$ m, and $x_{C1} = 280.668$ m, $y_{C1} = -60.887$ m.

In the coordinate system $\bar{x}_{(i)}, \bar{y}_{(i)}$ point A_1 (0, 0) should be connected to point C_1 , which has coordinates $\bar{x}_{C1} = \Delta\bar{x} = 348.263$ m, $\bar{y}_{C1} = \Delta\bar{y} = -193.064$ m ($\bar{s}_{A1} = 0$, $\bar{s}_{C1} = -0.17633$). The coefficients of polynomial (17) are as follows: $a_1 = -3.98915E-05$ 1/m², $a_2 = 1.67642E-07$ 1/m³, $a_3 = -1.90149E-10$ 1/m⁴. The determined connecting curve CC1 – compared to standard solutions and the connecting curves CC0, CC2, CC3, and CC4 – is shown in Figure 11 (marked in light blue).

The $\kappa(l)$ graph for the connecting curve CC1 is shown in Figure 10 (marked in light blue). The

maximum absolute value of curvature occurs at $l_0 = 57.484$ m and is $\kappa_{\max} = 0.0070038$ rad/m. In turn, the maximum value of $\Delta\kappa/\Delta l$ occurs at the beginning of the curve for $l \in \langle 0; 10.0001 \rangle$ m. Similar $V^{(a)}$ and $V^{(f)}$ were obtained for $h_0 = 65$ mm, i.e., $V^{(a)} = 48.575$ km/h and $V^{(f)} = 49.466$ km/h. Since Equation (7) gives $\Delta a = 0.225$ m/s², the speed $V^{(\psi)} = 79.834$ km/h. On curve CC1, the maximum speed $V_{\max} = 48$ km/h. It is lower compared to circular curves with transition curves.

The connecting curve was thus extended further, assuming $\Delta L_2 = 50$ m for curve CC2, $\Delta L_3 = 75$ m for curve CC3, and $\Delta L_4 = 100$ m for curve CC4. Table 6 presents the basic numerical values relating to the determination of the course of the connecting curves.

The connecting curves from Table 6 are shown in Figure 11 (alongside the standard solutions). The curvature graphs for these curves are presented in Figure 10. Based on the determined curvature (and the corresponding cant), train speeds on the given layout were determined, as illustrated in Table 7.

Table 6

Basic numerical values for determining connecting curves in Layout III

Connecting curve	$\Delta l(i)$ [m]	$\Delta\bar{x}$ [m]	$\Delta\bar{y}$ [m]	a_1 [1/m ²]	a_2 [1/m ³]	a_3 [1/m ⁴]
CC0	0	298.842	-188.723	-6.29471E-05	3.09547E-07	-4.10172E-10
CC1	25	348.263	-193.064	-3.98915E-05	1.67642E-07	-1.90149E-10
CC2	50	397.983	-197.405	-2.68843E-05	9.85532E-08	-9.76705E-11
CC3	75	447.503	-201.746	-1.89902E-05	6.16862E-08	-5.42588E-11
CC4	100	497.123	-206.087	-1.39209E-05	4.05691E-08	-3.20656E-11

[Author's own work].

Table 7

Calculation of speed for the determined connecting curves in Layout III

Connecting curve	$\Delta L(i)$ [m]	L_{CC} [m]	h_0 [mm]	$\kappa_{\max}$ [rad/m]	$V^{(a)}$ [km/h]	$\max(\Delta h/\Delta l)$ [‰]	$V^{(\theta)}$ [km/h]	$V^{(\psi)}$ [km/h]	$V_{\max}$ [km/h]
CC0	0	365.517	65	0.0091483	42.501	2.42418	41.581	72.623	41
CC1	25	409.801	65	0.0070038	48.575	2.03775	49.466	79.834	48
CC2	50	455.128	65	0.0055453	54.590	1.75425	57.461	92.735	54
CC3	75	501.247	70	0.0045066	61.327	1.53813	60.853	103.947	60
CC4	100	547.975	70	0.0037398	67.320	1.36817	68.412	113.733	67

[Author's own work].

In the standard solution (with transition curves), the speed $V_{\max} = 50$ km/h, and the length of the layout is 376.420 m. The use of a connecting curve with a length of 547.975 m permits an increase in speed to $V_{\max} = 67$ km/h. This does not necessarily have to be the upper limit for the increase in speed, provided, of course, that there are no objective constraints regarding the length of the connecting curve. The analysis of $W_1W_2 = 200$ m fully confirmed the earlier conclusions regarding $W_1W_2 = 400$ m.

6. Overall Evaluation of the Generated Solutions

The analysis conducted fully confirmed the thesis formulated in [16], namely that in the case of reverse curves, an appropriate distance should be maintained between the intersection points of adjacent main directions of the route – that is, this distance must not be too short. If this distance is significant for a given layout of main directions and makes it possible to solve the problem by using circular curves with transition curves, then reverse curves do not impose a significant limitation on the achievable train speed. This was demonstrated in the article in the case of Layout I, in which the distance between the main points was 1000 m. As demonstrated, the maximum speed achievable in that case was 132 km/h.

In the analogous Layout II, with the distance between main points limited to 400 m, the standard solution (i.e., using circular curves with transition curves) proved to be unacceptable due to the maximum achievable speed of only 77 km/h. Consequently, an attempt was made to shift away from the standard solution and to define a curve that directly connects the extreme main directions of the geometric layout. Appropriate differential equations with boundary conditions for the polynomial solution were used for this purpose. The practical usefulness of a model that

assumes zero curvature at the starting point and the endpoint has been demonstrated.

However, this is insufficient. The determined connecting curve must be extended further by appropriately selecting endpoints located on *Line 1* and *Line 3*. This article describes the procedure for achieving a train speed of 100 km/h. This required shifting point A on *Line 1* by 200 m, along with a corresponding shift of point C on *Line 3*.

In the third reverse-curve layout considered (i.e., Layout III), the distance between the main points was limited to 200 m. The use of circular curves with transition curves allows – in theory – for a speed of $V_{\max} = 50$ km/h. The speed is relatively low, and its practical application may be called into question due to design considerations. It would be difficult to accommodate rolling stock wheels on a track with a curve radius of less than 110 m.

Therefore, the only option is to use an extended connecting curve. This made it possible to reach a speed of 67 km/h while resolving the design-related issues. In this case, point A on *Line 1* and point C on *Line 3* were shifted by 100 m.

7. Conclusion

Conventional railway lines, which were mostly built in the 19th century and are still in operation today, continue to dominate the rail network (in terms of length). There is little doubt that these lines should be upgraded to meet contemporary operational requirements. This applies particularly to railway lines traversing difficult terrain (e.g., mountainous regions), where horizontal curves of small radius occur, often together with geometrically challenging track layouts such as compound curves and reverse curves. Improving the performance of these lines and increasing train operating speeds require appropriate modernization measures.

This paper investigates methods for modifying reverse curves, i.e., a geometric layout consisting of two

consecutive circular curves (usually with different radii), curving in opposite directions and directly connected to each other. These curves pose certain operational problems due to abrupt changes in curvature. This article builds on the issues addressed in [16]. The study in question presents conventional solutions involving the creation of a geometric layout consisting of two circular arcs oriented in opposite directions and four transition curves connecting these curves to the corresponding main directions of the route. However, attention was drawn to the limitations associated with the use of the proposed method. The goal was to maintain an appropriate distance between the intersection points of adjacent main directions (this distance cannot be too short). On modernized railway lines, this requirement is frequently not satisfied, making it necessary to modify reverse curves accordingly.

The article considered three geometric layouts of reverse curves. In Layout I, there was a considerable distance between the main points of the route (i.e., 1000 m). It has been shown that, in such cases, a standard solution involving the use of circular curves with transition curves can be successfully applied. When there is a significant distance between the main points along the route, reverse curves do not significantly limit train speeds.

In Layout II there was a considerable distance between the main points of the route (i.e., 400 m). From the perspective of train speeds, the standard solution did not meet the requirements in this case. Therefore, a curve connecting the two extreme main directions was proposed; this curve satisfies the appropriate boundary conditions and the assumed differential equation. By increasing the length of this curve, higher maximum train speeds can be achieved. One limitation may be the obstacles that arise, for example, due to the terrain conditions.

Layout III is characterized by a short distance between the main points of the route (i.e., 200 m). The standard solution also proves unacceptable here for structural reasons, i.e., the incorporation of rolling stock wheels into the track with very small radii of the circular curves. Using an extended connecting curve may prove to be the only advantageous solution in this situation.

The proposed method of modifying reverse curves provides a practical solution to design problems. This is particularly important for railway lines running through difficult terrain, where the distances between the main points of a reverse curve are often short. This article outlines the best course of action in such situations.

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